

A DOUBLING INTEGRAL FOR G_2

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CONTENTS

1. Introduction	1
2. The Global Integral	3
2.1. Eisenstein Series	4
2.2. Fourier Coefficient	4
2.3. Global Integral	5
2.4. Main Global Result	6
3. The Unramified Computations	12
3.1. Notation	12
3.2. Main Local Result	14
3.3. Transformation of the integral	16
3.4. Proof of Main Unramified Identity	17
4. The normalizing factor of the Eisenstein series	20
5. Calculation of $I(s, t)$	21
5.1. First reduction	22
5.2. Second Reduction	26
5.3. An SL_5 period	27
5.4. Final calculation of $I(s, t)$	31
References	33

1. INTRODUCTION

In this paper we introduce a new global integral which represents the standard L function attached to a cuspidal representation of the exceptional group $G_2(\mathbb{A})$. Here \mathbb{A} is the adèle ring of a global field F . In [PS-R] the authors introduced a global integral which represents

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the standard L function for a classical group H . To describe their construction, for $i = 1, 2$, let σ_i denote two cuspidal representations of $H(\mathbb{A})$. Their global integral is given by

$$\int_{H(F) \times H(F) \backslash H(\mathbb{A}) \times H(\mathbb{A})} \varphi_{\sigma_1}(h_1) \varphi_{\sigma_2}(h_2) E((h_1, h_2), s) dh_1 dh_2$$

Here $E(\cdot, s)$ is a certain Eisenstein series. Unfolding the integral, one obtains the bilinear form

$$\langle \pi_1(h) \varphi_{\sigma_1}, \varphi_{\sigma_2} \rangle = \int_{H(F) \backslash H(\mathbb{A})} \varphi_{\sigma_1}(h_1 h) \varphi_{\sigma_2}(h_1) dh_1$$

as inner integration. Using that, the authors established the fact that the global integral is Eulerian. Moreover, assuming that σ_1 and σ_2 are contragredient, then this integral is nonzero for some choice of data. Hence, the above global integral is nonzero for *all* cuspidal representation σ_1 of $H(\mathbb{A})$. Integrals of this type are now known as “doubling integrals.”

In this paper we construct a doubling integral which represents the standard L function for $G_2(\mathbb{A})$. The global integral we construct uses a certain Eisenstein series defined on the exceptional group $E_8(\mathbb{A})$. It is introduced in section 2 integral (3). Thus, as in the classical groups, we obtain an integral construction which represents the standard L function, and which is not zero for *all* cuspidal representations. By attaching a character χ to the Eisenstein series, our construction actually represents the twisted L function.

A global construction for this L function is given in [G1], which is valid only for generic cuspidal representations. Another construction for this L function, which is valid also for cuspidal representations which are not generic, is given in [S]. In contrast to our construction and to [G1], this global integral unfolds to a non-unique model.

In this paper we prove the very basic properties of our construction. In section 2, after some preparation we introduce the global integral, unfold it, and show that it is Eulerian. In the third section, using MacDonald’s formula, we carry out the unramified computations. In sections 4 and 5 we carry out some computations which we use in the proof of the unramified calculations. For some of the calculations, we used the software LiE [L] and another software program [E], written by the second named author. This is mainly due to the sheer size of the calculations, owing to the large number of roots in E_8 , the size of the Weyl group, etc., as well as to the large number of such calculations which must be performed. It is worth noting that any individual calculation which was performed with software can also be checked by hand.

We also mention, that as in [PS-R] pages 50-51, the local unramified computation gives us a definition of a generating function for the standard L function. Thus, it follows from section 3 equation (16), that if we define $N(s)$ to be the normalizing factor of the Eisenstein

series (cf. section 4) and we define

$$\Delta_s(g) = N(s) \int_{U_0(F)} f(w_0zu(1, g), s\psi_U(u)du$$

then

$$\int_{G_2(F)} \omega_\pi(g) \Delta_s(g) dg = L(\pi, s)$$

We expect that our global construction will have certain applications in the study of this specific L function, and its twists by a character χ . The first, is the study of the poles of this L function. The construction in [G1] implies that for generic cuspidal representations, this L function can have at most a simple pole. As follows from section 4, one expects that the Eisenstein series we use would have at most double poles. In fact one expects that there are cuspidal representations, which are CAP with respect to the Borel subgroup, whose L functions will have a double pole. Such CAP representations were constructed in [G-G-J].

In the near future we hope to prove the following

Conjecture 1. *The twisted partial standard L function $L^S(\pi \otimes \chi, s)$ can have at most a double pole.*

2. THE GLOBAL INTEGRAL

In this section we introduce the global integral, and carry out the unfolding process. We work with the unique split F -group of type E_8 , which we assume to be equipped with a choice of maximal torus T and Borel subgroup $B = TU_{\max}$. Here U_{\max} is a maximal unipotent subgroup of E_8 . For $H \subset E_8$ a T -stable subgroup, $\Phi(H, T)$ is the set of roots of T in H . Also for any reductive group H with maximal torus S , $W(H, S)$ is the Weyl group of H relative to S . For $\Phi(E_8, T)$ and $W(E_8, T)$, we may write Φ and W respectively. In this paper we shall label the roots of E_8 by α_i , ($1 \leq i \leq 8$). The labeling we use is as in [G-S]. Let w_i denote the simple reflection corresponding to the root α_i . We shall denote the product $w_{i_1} \dots w_{i_n}$ by $w[i_1 \dots i_n]$. Write U_α for the one-dimensional unipotent subgroup attached to the root α , and equip G with a realization $\{x_\alpha : \mathbb{G}_a \rightarrow U_\alpha \mid \alpha \in \Phi(E_8, T)\}$, consisting of an isomorphism $\mathbb{G}_a \rightarrow U_\alpha$ for each root α . We assume that the structure constants are determined as in [G-S]. For $1 \leq i \leq 8$, the product $x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1)$ is a representative for the simple reflection w_i . This, in turn, determines a standard representative in $G(F)$ for any word in the simple reflections. We shall often abuse notation by conflating this representative with the Weyl word it represents.

2.1. Eisenstein Series. For $1 \leq i \leq 8$ we let P_i denote the standard maximal parabolic subgroup of E_8 such that α_i is a root of the unipotent radical and the remaining simple roots of E_8 are roots of the standard Levi factor. We consider the group P_2 . Its Levi factor, M_2 , is isomorphic to $\{g \in GL_8 : \det g \text{ is a square}\}$. The group M_2 has a rational character whose square is \det . (Indeed, M_2 acts on the highest weight vectors in the second fundamental representation of E_8 by such a representation.) Denote this rational character by $\det^{\frac{1}{2}}$. The modular quasicharacter δ_{P_2} of P_2 is the seventeenth power of $\det^{\frac{1}{2}}$.

Let χ be a character of \mathbb{A}^\times trivial on F^\times . Regard χ as a representation of $M_2(\mathbb{A})$ by composing with $\det^{\frac{1}{2}}$. Consider the induced representation $Ind_{P_2(\mathbb{A})}^{E_8(\mathbb{A})} \delta_{P_2}^s \chi$. This representation has an automorphic realization as a space of Eisenstein series. Specifically, for $f_{s,\chi} \in Ind_{P_2(\mathbb{A})}^{E_8(\mathbb{A})} \delta_{P_2}^s \chi$ the corresponding Eisenstein series is defined by $E(h, f_{s,\chi}) := \sum_{\gamma \in P_2(F) \backslash E_8(F)} f_{s,\chi}(\gamma g)$ for $\text{Re}(s)$ large and by meromorphic continuation elsewhere.

2.2. Fourier Coefficient. Next we describe a Fourier coefficient which plays an important role in this construction. As explained in [G2] one can associate with every unipotent orbit of a given group which defined over an algebraic closed field, a set of Fourier coefficients. In [G2] it is explained how to do it in the classical groups, but it is similar in the exceptional groups. The Fourier coefficient we will define is attached to the unipotent orbit $2A_2$ of E_8 .

In the notation fixed above, P_1 denotes the standard maximal parabolic subgroup of E_8 whose Levi factor M_1 is the product of a derived subgroup isomorphic to $Spin_{14}$ and a one-dimensional torus. Let U denote its unipotent radical of P_1 . Then U is a two step unipotent group whose dimension is 78. The center $Z(U)$ is 14 dimensional and can be identified with the standard (“vector”) representation of $Spin_{14}$. The quotient $U/Z(U)$ is 64 dimensional and can be identified with a half-spin representation of $Spin_{14}$.

Let ψ denote a nontrivial additive character of the group $F \backslash \mathbb{A}$. The choice of ψ identifies F with the Pontriagin dual of $F \backslash \mathbb{A}$. Characters of $U(\mathbb{A})$ trivial on $U(F)$ are then identified with the F -points of the rational representation of M_1 which is dual to $U/Z(U)$. This would correspond to the other half-spin representation of $Spin_{14}$.

Over an algebraically closed field, this representation of M_1 has an open orbit. As recorded in the tables on p. 405 of [C] and p.200 of [Ka], the identity component is isomorphic to $G_2 \times G_2$. Moreover, using the discussion on p. 267 of [Ki], it’s not hard to show that the number of components in the stabilizer is two. We now choose a character of $U(F) \backslash U(\mathbb{A})$ which corresponds to a point in general position.

Define the character ψ_U of $U(F) \backslash U(\mathbb{A})$ as follows. Using the fixed realization $\{x_\alpha\}$, write $u \in U(\mathbb{A})$ as $u = x_{11221111}(r_1)x_{11122111}(r_2)x_{12232210}(r_3)x_{11233210}(r_4)u'$, where u' is a product of elements $x_\alpha(u_\alpha)$ corresponding to roots α which are not among the four listed above. Then

we define $\psi_U(u) = \psi(r_1 + r_2 + r_3 + r_4)$. It's not difficult to calculate the identity component of the stabilizer of this character, which turns out to be isomorphic to $G_2 \times G_2$, proving that the character is indeed in general position.

To be precise, the images of

(1)

$$x_{\pm 00010000}(r); x_{\pm 01000000}(r)x_{\pm 00100000}(r)x_{\pm 00001000}(-r); x_{\pm 01010000}(r)x_{\pm 00110000}(r)x_{\pm 00011000}(r)$$

$$x_{\pm 01011000}(r)x_{\pm 01110000}(r)x_{\pm 00111000}(-r); x_{\pm 01111000}(r); x_{\pm 01121000}(r)$$

generate a subgroup of E_8 which is isomorphic to G_2 . One may check that an element of this group which normalizes/centralizes its maximal torus also normalizes/centralizes the full maximal torus of E_8 . This induces an embedding of the Weyl group of this copy of G_2 into that of E_8 . The images of the two simple reflections are w_4 and $w[235]$.

A second copy of G_2 is generated by the images of the following homomorphisms:

(2)

$$x_{\pm 00000010}(r); x_{\pm 00000001}(r)x_{\pm 01011100}(-r)x_{\pm 00111100}(r); x_{\pm 00000011}(r)x_{\pm 01011110}(r)x_{\pm 001111100}(-r)$$

$$x_{\pm 01122210}(r)x_{\pm 01011111}(r)x_{\pm 00111111}(-r); x_{\pm 01122211}(r); x_{\pm 01122221}(r).$$

In this case, the simple reflections in the Weyl group map to w_7 and $w[865423456]$ in the Weyl group of E_8 .

Finally, suitable representatives for the Weyl element $w[657486576]$ stabilize ψ_U and act on the two copies of G_2 by reversing them.

We shall denote the stabilizer of ψ_U by H^\pm and its identity component by H . Note that $T \cap H$ and $U_{\max} \cap H$ are a maximal torus and maximal unipotent subgroup of H , respectively. The one parameter subgroups listed above equip each copy of G_2 with a realization and the structure constants of the two realizations. Thus, by equipping G_2 (regarded as an abstract group) with a realization having the same structure constants, we pin down a specific identification of H with $G_2 \times G_2$.

One may now form the Fourier coefficient mapping

$$\varphi \mapsto \varphi^{(U, \psi_U)}(h) := \int_{U(F) \backslash U(\mathbb{A})} \varphi(uh) \psi_U(u) du, \quad (\varphi \in C^\infty(HU(F) \backslash E_8(\mathbb{A})))$$

with image in the space $C^\infty(H(F)U(\mathbb{A})_{\psi_U} \backslash E_8(\mathbb{A}))$, of smooth, left $H(F)$ -invariant, $(U(\mathbb{A}), \psi_U)$ -equivariant functions $E_8(\mathbb{A}) \rightarrow \mathbb{C}$.

2.3. Global Integral. In order to define a global integral, we apply the Fourier coefficient mapping defined in section 2.2 to the Eisenstein series of section 2.1. The result is a smooth function of uniformly moderate growth $H(F) \backslash H(\mathbb{A}) \rightarrow \mathbb{C}$. It may therefore be integrated

over $H(F)\backslash H(\mathbb{A})$ against a cusp form. Thus, let $\pi = \pi_1 \otimes \pi_2$ denote an irreducible cuspidal representation of $H(\mathbb{A})$.

The global integral we consider is

$$(3) \quad \int_{G_2(F)\backslash G_2(\mathbb{A})} \int_{G_2(F)\backslash G_2(\mathbb{A})} \varphi_{\pi_1}(g_1) \varphi_{\pi_2}(g_2) \int_{U(F)\backslash U(\mathbb{A})} E(u(g_1, g_2), f_{s,\chi}) \psi_U(u) du dg_1 dg_2$$

Here $G_2 \times G_2$ is embedded in E_8 as the stabilizer of the character ψ_U as described above. Also, φ_{π_1} and φ_{π_2} are vectors in the space of π_1 and π_2 respectively.

2.4. Main Global Result. The main result of this section is the following

Theorem 1. *Let w_{lng} denote the shortest element of $W(M_2, T)w_\ell W(M_1, T)$, where w_ℓ is the longest element of W . Let $\nu_0 = w[345678243546576]$, and let $w_0 = w_{\text{lng}}\nu_0$. Let $U_0 = U \cap w_0^{-1}\overline{U}_{\max}w_0 = \prod_{\alpha \in \Phi(U, T): w_0\alpha < 0} U_\alpha$. Denote*

$$\langle \pi_1(g) \varphi_{\pi_1}, \varphi_{\pi_2} \rangle = \int_{G_2(F)\backslash G_2(\mathbb{A})} \varphi_{\pi_1}(g_1 g) \varphi_{\pi_2}(g_1) dg_1,$$

and let $z = x_{00011100}(1)x_{00001110}(1)x_{00000111}(-1)$. Then the integral (3) is equal to

$$(4) \quad \int_{G_2(\mathbb{A})} \int_{U_0(\mathbb{A})} \langle \pi_1(g) \varphi_{\pi_1}, \varphi_{\pi_2} \rangle f_{s,\chi}(w_0 z u(g, 1)) \psi_U(u) du dg.$$

for all $\text{Re}(s)$ large.

Proof. Throughout this proof, we assume s lies in the domain of absolute convergence for $E(f_{s,\chi}, g)$. For $\gamma_0 \in E_8(F)$ define

$$E_{\gamma_0}(g, f_{s,\chi}) := \sum_{\gamma \in (\gamma_0^{-1} P_2 \gamma_0 \cap UH)(F)\backslash UH(F)} f_{s,\chi}(\gamma_0 \gamma g).$$

Clearly, the sum is absolutely convergent. Moreover $E_{\gamma_0}(g, f_{s,\chi})$ is left $UH(F)$ -invariant for each γ_0 and

$$E(g, f_{s,\chi}) = \sum_{\gamma_0 \in P_2(F)\backslash E_8(F)/UH(F)} E_{\gamma_0}(g, f_{s,\chi}).$$

Since E_{γ_0} is $UH(F)$ -invariant, we can consider its Fourier coefficient $E_{\gamma_0}^{(U, \psi_U)}$.

Rather than look for an exact set of representatives for the double cosets $P_2(F)\backslash E_8(F)/UH(F)$ it is convenient to work with a larger subset of $E_8(F)$ which clearly contains a set of representatives. Write $L_{4,7}$ for the subgroup of E_8 generated by $U_{\pm\alpha_4}, U_{\pm\alpha_7}$. Clearly, $L_{4,7} \subset H$. Let $\overline{U}_{\max} = \prod_{\alpha < 0} U_\alpha$ denote the maximal unipotent subgroup of E_8 opposite U_{\max} and for $w \in W$ let $N_w = U_{\max} \cap w^{-1}\overline{U}_{\max}w = \prod_{\alpha > 0, w\alpha < 0} U_\alpha$. Let

$$\dot{W}(M_2, L_{4,7}) = \{\sigma \in W(E_8, T) : \sigma \text{ is of minimal length in } W(M_2, T) \cdot \nu \cdot \langle w[4], w[7] \rangle \sigma\}.$$

Here $\langle w[4], w[7] \rangle$ denotes the subgroup of $W(E_8, T)$ generated by the $w[4]$ and $w[7]$. Then it follows from the Bruhat decomposition that the set

$$(5) \quad \{\sigma\delta \mid \sigma \in \dot{W}(M_2, L_{4,7}) \ \delta \in M_1 \cap N_\sigma(F)\}.$$

is a set of double coset representatives for $P_2(F) \backslash E_8(F) / U(F) L_{4,7}(F)$, and hence surjects onto $P_2(F) \backslash E_8(F) / U(F) H(F)$.

We say that $w \in W$ is left M_2 reduced if it is the shortest element of $W(M_2, T) \cdot w$. Assume that $\gamma_0 = w\mu$, with $w \in W$ left M_2 reduced and $\mu \in M_1(F)$. This certainly includes the case $\gamma_0 = \sigma\delta$ as in (5). Then it can be shown that $\gamma_0 u h \gamma_0^{-1} \in P_2$ if and only if $\gamma_0 u \gamma_0^{-1} \in P_2$ and $\gamma_0 h \gamma_0^{-1} \in P_2$. It follows that

$$(6) \quad \begin{aligned} E_{\gamma_0}^{(U, \psi_U)}(h, f_{s, \chi}) &= \int_{U\gamma_0(F) \backslash U(\mathbb{A})} \sum_{\gamma \in (H \cap \gamma_0^{-1} P_2 \gamma_0)(F) \backslash H(F)} f_{s, \chi}(\gamma_0 \gamma u h) \psi_U(u) du \\ &= \int_{U^w(F) \backslash U(\mathbb{A})} \sum_{\gamma \in (H \cap \gamma_0^{-1} P_2 \gamma_0)(F) \backslash H(F)} f_{s, \chi}(w u \mu \gamma h) [\mu \cdot \psi_U](u) du. \end{aligned}$$

where, $U^w := U \cap w^{-1} P_2 w$, and $[\mu \cdot \psi_U](u) := \psi_U(\mu^{-1} u \mu)$ for $\mu \in M_1(F)$, $u \in U(\mathbb{A})$.

Lemma 1. *Take $w \in W$ left M_2 reduced, and $\mu \in M_1(F)$.*

- (1) *If the character $\mu \cdot \psi_U$ is nontrivial on the group $U^w(\mathbb{A})$, then $E_{w\mu}^{(U, \psi_U)} = 0$.*
- (2) *Set $\gamma_0 = w\mu$. If the group $(H \cap \gamma_0^{-1} P_2 \gamma_0)$ contains the unipotent radical of a parabolic subgroup of H , then $E_{\gamma_0}^{(U, \psi_U)}$ is orthogonal to cuspforms $H(F) \backslash H(\mathbb{A}) \rightarrow \mathbb{C}$.*
- (3) *For any $\gamma_0 \in E_8(F)$, if the function $E_{\gamma_0}^{(U, \psi_U)}$ is zero (resp. orthogonal to cuspforms), then $E_{\gamma'_0}^{(U, \psi_U)}$ is zero (resp. orthogonal to cuspforms) for all $\gamma'_0 \in P_2(F) \cdot \gamma_0 \cdot U(F) H^\pm(F)$.*

Remark 1. “Orthogonal to cuspforms” just means that the integral

$$(7) \quad J_{\gamma_0}(f_{s, \chi}, \varphi) := \int_{H(F) \backslash H(\mathbb{A})} E_{\gamma_0}^{(U, \psi_U)}(h, f_{s, \chi}) \varphi(h) dh,$$

is zero for any $f_{s, \chi} \in \text{Ind}_{P_1(\mathbb{A})}^{E_8(\mathbb{A})} \delta_{P_2}^s \chi$ and any cuspform $\varphi : H(F) \backslash H(\mathbb{A}) \rightarrow \mathbb{C}$. Note that (7) is precisely the contribution of E_{γ_0} to our global integral. The function $E_{\gamma_0}^{(U, \psi_U)}(f_{s, \chi})$ need not be L^2 , so strictly speaking there is no inner product space here. However, it is of uniformly moderate growth, so it follows from the decay properties of cuspforms that (7) is absolutely convergent.

Proof. (1) The function $g \mapsto f_{s, \chi}(wg)$ is left-invariant by $U^w(\mathbb{A})$. So, (6) can be written as a double integral with the inner integration being

$$\int_{U^w(F) \backslash U^w(\mathbb{A})} [\mu \cdot \psi_U](u) du.$$

- (2) The group $H \cap \gamma_0^{-1}P_2\gamma_0$ normalizes U and also the subgroup $U \cap \gamma_0^{-1}P_2\gamma_0$. Moreover, $H(\mathbb{A})$ stabilizes ψ_U . The function $f_{s,\chi}$ is left-invariant, by the \mathbb{A} -points of any unipotent subgroup of P_2 . Consequently, the function

$$h \mapsto \int_{U^w(F) \backslash U(\mathbb{A})} f_{s,\chi}(\gamma_0 u h) \psi_U(u) du$$

is left-invariant by the \mathbb{A} -points of any unipotent subgroup of $H \cap \gamma_0^{-1}P_2\gamma_0$. The second part follows.

- (3) It follows from the definition of E_{γ_0} that $E_{p\gamma_0 h} = E_{\gamma_0}$ for $p \in P_2(F)$ and $h \in H(F)$. For $h^- \in H^\pm(F) \setminus H(F)$, one has $E_{\gamma_0 h^-}(g, f_{s,\chi}) = E_{\gamma_0}(h^- g, f_{s,\chi})$ for all $g \in E_8(\mathbb{A})$. \square

Lemma 2. *Let*

$$(8) \quad \text{Supp}_\Phi(\psi_U) = \{11221111, 11122111, 12232210, 11233210\}.$$

Take $\gamma_0 = \sigma\delta$ as in (5). If $\sigma\alpha > 0$ for some $\alpha \in \text{Supp}_\Phi(\psi_U)$, then $E_{\gamma_0}(f_{s,\chi}, \varphi) = 0$.

Proof. If $\alpha \in \text{Supp}_\Phi(\psi)$ the ψ_U is nontrivial on U_α . The condition $\sigma\alpha > 0$ implies that $U_\alpha \subset U^\sigma$, and ensures that ψ_U is nontrivial on U^σ . What must be shown is that $[\delta \cdot \psi_U]$ remains nontrivial on $U_\alpha(\mathbb{A})$, for all $\delta \in N_\nu(F)$. This follows from the fact that N_ν is contained in the standard maximal unipotent and no two elements of $\text{Supp}_\Phi(\psi_U)$ differ by a positive root. \square

Proposition 1. *The set $\dot{W}(M_2, L_{4,7})$ as in (5) has 6576 elements, of which all but 25 map at least one of the four roots listed in lemma 2 to a positive root.*

Proof. One can check this using the computer package LiE [L]. \square

Let $S = \{\sigma \in \dot{W}(M_2, L_{4,7}) : \sigma\alpha > 0 \quad \forall \alpha \in \text{Supp}_\Phi(\psi)\}$. According to proposition 1, S has 25 elements. Now, for $\sigma \in \dot{W}(M_2, L_{4,7})$ and $\delta \in (N_\sigma \cap M)(F)$, one has $P_2(F)\sigma\delta L_{4,7}(F) \subset P_2(F)\sigma\delta UH(F) \subset P_2(F)\sigma\delta P_1(F)$. So, it makes sense to sort these 25 elements according to the image in $P_2(F)\sigma\delta P_1(F)$. Another straightforward computer check using [L] yields the next lemma.

Lemma 3. *If $\sigma \in S$ then $P_2\sigma P_1$ contains either*

$$(9) \quad w_{\text{sht}} := w[2431542345654234576542314354287654231435426543765428765431]$$

or

$$(10)$$

$$w_{\text{lng}} := w[24315423456542314354276542314354265437654287654231435426543765428765431].$$

If $S_\star = \{\sigma \in S : w_\star \in P_2\sigma P_1\}$, ($\star = \text{sht}, \text{lng}$), then S_{sht} has 9 elements and S_{lng} has 16. The 9 elements of S_{sht} are all in the same $P_2(F), H(F)$ double coset, and the shortest of them is $w_{\text{sht}}w[56]$.

Proposition 2. *For each $\sigma \in S_{\text{sht}}$, the restriction of $\delta \cdot \psi_U$ to U^σ is trivial if and only if $\delta \in H(F)$.*

Proof. One can check this on a case-by-case basis using the program “DCA3” from the egut package [E]. \square

Corollary 1. *If $\sigma \in S_{\text{sht}}$ then $E_{\sigma\delta}^{(U, \psi_U)}$ is orthogonal to cuspforms.*

Proof. We use all three parts of lemma 1. It is clear that $E_{w_{\text{sht}}w[56]}^{(U, \psi_U)}$ is orthogonal to cuspforms by part (2). Hence the same holds for any element of $P_2(F)w_{\text{sht}}H(F)$ by part (3), and the proposition shows that $E_{\sigma\delta}^{(U, \psi_U)}$ is identically zero the rest of the time by part (1). \square

Lemma 4. *Suppose that $\sigma \in S_{\text{lng}}$ and $w_{\text{lng}}^{-1}\sigma$ can be written as a word in the simple reflections without using $w[4]$. Then $E_{\sigma\delta}^{(U, \psi_U)}$ is orthogonal to cuspforms for all $\delta \in N_\sigma \cap M_1(F)$. The same is true with “4” replaced by “7”.*

Proof. Set $\nu = w_{\text{lng}}^{-1}\sigma$. Since w_{lng} is left M_1 -reduced, it follows that $\sigma\alpha < 0 \iff \nu\alpha < 0$ for $\alpha \in \Phi(M_1, T)$. From this it follows that $N_\sigma \cap M_1 = N_\nu$.

Let $P_4^{M_1} := M_1 \cap P_4$, i.e., the maximal standard parabolic subgroup of M_1 obtained by intersecting M_1 with the standard maximal parabolic subgroup P_4 of G . Thus, the only simple root α of M_1 such that U_α is contained in the unipotent radical of $P_4^{M_1}$ is α_4 . Suppose that ν can be expressed as a word in the simple reflections without using 4. Then ν , is actually in the Levi $M_4^{M_1}$ of $P_4^{M_1}$, and, consequently, so is N_ν . Clearly, the unipotent radical $U_4^{M_1}$ of $P_4^{M_1}$ is contained in Q_w , and hence in $\delta^{-1}\nu^{-1}Q_w\nu\delta$ for any $\delta \in N_\nu$. Since the intersection of $U_4^{M_1}$ with H is the unipotent radical of a parabolic subgroup of H . The lemma follows from lemma 1, part (2). The previous argument also works if “4” is replaced by “7” throughout, which completes the proof. \square

Inspecting the 16 elements of S_{lng} yields a reduction.

Lemma 5. *Let $S'_{\text{lng}} = \{\sigma \in S_{\text{lng}} \mid w_{\text{lng}}^{-1}\sigma \text{ contains a 4 and a 7}\}$ then S'_{lng} has 8 elements.*

Proposition 3. *Let $w_0 = w_{\text{lng}}\nu_0$ where $\nu_0 = w[345678243546576]$. For three of the eight elements $\sigma \in S'_{\text{lng}}$, the function $E_{\sigma\delta}^{(U, \psi_U)}$ is orthogonal to cuspforms for all $\delta \in N_\sigma \cap M_1$. The remaining five have the property that $P_2\sigma(N_\sigma \cap M)$ intersects $P_2w_0N_{w_0}H^\pm$, and $\delta \cdot \psi_U$ is nontrivial on U^σ unless $\sigma\delta \in P_2(F)w_0N_{w_0}(F)H^\pm(F)$.*

Proof. This can be checked on a case by case basis using the egut program DCA3. \square

The results of the previous section imply that $E_{\gamma_0}^{(U, \psi_U)}$ is orthogonal to cuspforms whenever $P_2(F)\gamma_0 H^\pm \cap w_0(N_{w_0} \cap M_1)(F)$ is empty. In this section we must study $E_{\gamma_0}^{(U, \psi_U)}$ for $\gamma_0 \in w_0 N_{w_0}(F)$. Recall that $w_0 = w_{\text{lng}} \nu_0$ where $\nu_0 = w[345678243546576]$. And that $N_{w_0} \cap M_1$ can be described more simply as N_{ν_0} .

We have

$$\Phi(N_{\nu_0}, T) = \left\{ \begin{array}{l} 00000100, 00000110, 00000111, 00001100, 00001110, 00001111, 00011100, 00011110, \\ 00011111, 00111100, 00111110, 00111111, 01122210, 01122211, 01122221 \end{array} \right\}.$$

Write $\delta \in N_{\nu_0}$ as $\prod_{\alpha \in \Phi(N_{\nu_0}, T)} x_\alpha(\delta_\alpha)$ with the product taken in the order the roots are listed above. Since H contains every element of E_8 of the form

$$\begin{aligned} & x_{00000001}(r)x_{01011100}(-r)x_{00111100}(r), & x_{00000011}(r)x_{00111110}(-r)x_{01011110}(r), \\ & x_{01011111}(r)x_{01122210}(r)x_{00111111}(-r), & x_{01122211}(r), \quad \text{or} \quad x_{01122221}(r), \end{aligned}$$

It follows that every element of $\sigma N_{\nu_0}(F)$ lies in the same $P_2(F), H(F)$ -double coset as $\nu_0 \delta$ for some δ such that

$$(11) \quad \delta_{00111100} = \delta_{00111110} = \delta_{01122210} = \delta_{01122211} = \delta_{01122221} = 0.$$

For such δ , the condition $\delta \cdot \psi_U|_{U^\sigma} \equiv 1$ implies

$$\delta_{00011111} = \delta_{00001111} = \delta_{00000110} = \delta_{00000100} = \delta_{00111111} = 0$$

$$(12) \quad \delta_{00000111} = \delta_{00001100}\delta_{00011110} - \delta_{00001110}\delta_{00011100}$$

Lemma 6. *The set*

$$\{x_{00001100}(\delta_{00001100})x_{00011100}(\delta_{00011100})x_{00001110}(\delta_{00001110})x_{00000111}(\delta_{00000111})x_{00011110}(\delta_{00011110})\}$$

is a four-dimensional abelian subgroup $D_0 \subset N_\nu$, which is normalized by

$$(13) \quad M_0 := (T \cap H) \cdot SL_2^{\alpha_4} \cdot SL_2^{\alpha_7} \subset H \cap \nu^{-1}Q_w \nu.$$

The subset consisting of elements which satisfy (12) is a union of three orbits for this action, represented by the identity, $x_{00011110}(1)$ and

$$x_{00011100}(1)x_{00001110}(1)x_{00000111}(-1).$$

Proof. First, $\nu \cdot \alpha_4 = \alpha_7$, while $\nu \cdot \alpha_7 = \alpha_4$. Both lie in $\Phi(L_w, T)$, and this proves that $M_0 \subset H \cap \nu^{-1}Q_w \nu$. Verifying that the set D_0 is indeed a subgroup is as simple as checking that no two of the roots

$$(14) \quad 00001100, 00011100, 00001110, 00000111, 00011110$$

sum to give another root of E_8 . It is obvious that T normalizes D_0 . Proving that $SL_2^{\alpha_4} \cdot SL_2^{\alpha_7}$ does as well is as simple as verifying that when either α_4 or α_7 is subtracted from a root listed in (14), the result is either not a root, or one of the other roots listed in (14).

The action of M_0 on D_0 can be identified with the action of GL_2^2 on $\mathbb{G}_a \times \text{Mat}_2$ via

$$(15) \quad (g_1, g_2) \cdot (x, X) = (\det g_1 \det g_2^{-1} x, g_1 X g_2^{-1}),$$

in such a way that the subvariety defined by (12) corresponds to that defined by $x = -\det X$. It's clear that

- this subset is preserved by the action (15),
- it is the union of three orbits, namely, the trivial orbit, $\{(0, 0)\}$, $\{(0, X) : X \neq 0, \det X = 0\}$ and $\{(-\det X, X) : \det X \neq 0\}$.
- the elements given above do indeed represent these orbits.

□

Corollary 2. *If $\gamma_0 \notin P_2(F)w_0zH^\pm(F)$, then $E_{\gamma_0}^{(U, \psi_U)}$ is orthogonal to cuspforms.*

Proof. Indeed, one can check using LiE that $w_0\alpha_i > 0$ for $i = 2, 3, 4, 5$. Hence $w_0^{-1}Hw_0$ contains the full unipotent radical of the copy of G_2 generated by (1).

Let G_1 denote the subgroup of E_8 , isomorphic to $Spin_8$, generated by $U_{\pm\alpha_i}$ for $i = 2, 3, 4, 5$. Consider the maximal parabolic subgroup of G_1 whose unipotent radical contains U_{α_4} . Let V_4 denote this unipotent radical. Then $x_{00011110}(1)V_4x_{00011110}(-1) \subset V_4U_{01121110}U_{01122110}$, and $w_0\alpha > 0$ for each $\alpha \in \{01121110, 01122110\} \cup \Phi(V_4, T)$. Hence $w_0x_{00011110}(1)$ conjugates $(V_4 \cap H)$ into P_2 . The result follows from lemma 1, part (2). □

This completes the proof that the global integral (3) is equal to $J_{\gamma_0}(f_{s, \chi}, \varphi_{\pi_1} \cdot \varphi_{\pi_2})$, defined by (7), for $\gamma_0 = w_0z$.

Remark 2. *For each $\gamma_0 \in E_8(F)$, J_{γ_0} is a bilinear form between $\text{Ind}_{P_2(\mathbb{A})}^{E_8(\mathbb{A})} \delta_P^s \chi$ and the space of cuspforms on $H(F) \backslash H(\mathbb{A})$. Define $\text{Supp}(J)$ to be the set of $\gamma_0 \in E_8(F)$ such that J_{γ_0} is nonzero. Then we have shown that $\text{Supp}(J)$ is a union of $P_2(F), UH^\pm(F)$ double cosets, and that it vanishes off of a single $P_2(F), UH$ -double coset. It is therefore worth asking whether $P_2(F)w_0zUH^\pm(F) = P_2(F)w_0zUH(F)$, for if this is not so, then we have proved that J_{γ_0} vanishes identically. Thankfully, the answer is yes. Indeed, if we define ${}^tz = x_{-00011100}(1)x_{-00001110}(1)x_{00000111}(-1)$, then $z^{-1}{}^tz z^{-1}$ is a representative for the Weyl element $w[657486576]$. Recall that certain representatives for this Weyl element lie in $H^\pm(F) \backslash H(F)$. Moreover one can choose $t_0 \in T(F)$ such that $z^{-1}{}^tz z^{-1}t_0 \in H^\pm(F) \backslash H(F)$ and $t_0^{-1}z^{-1}t_0 = z$. As $w_0{}^tz t_0 w_0 \in P_2(F)$, it follows that $z \cdot (z^{-1}{}^tz z^{-1}t_0)$ and z are in the same $P_2(F), H(F)$ double coset.*

Let J_{γ_0} be defined as in (7). Then

$$J_{\gamma_0}(f_{s,\chi}, \varphi) = \int_{(H \cap \gamma_0^{-1} P_2 \gamma_0)(F) \backslash H(\mathbb{A})} \varphi(h) \int_{U^{\gamma_0}(F) \backslash U(\mathbb{A})} f_{s,\chi}(\gamma_0 u h) \psi_U(u) du dh.$$

We have shown that the global integral (3) is equal to $J_{w_0 z}(f_{s,\chi}, \varphi_{\pi_1} \varphi_{\pi_2})$.

Lemma 7. *Define w_0, z and U_0 as in theorem 1. Then*

$$\int_{U^{w_0 z}(F) \backslash U(\mathbb{A})} f_{s,\chi}(w_0 z u h) \psi_U(u) du = \int_{U_0(\mathbb{A})} f_{s,\chi}(\gamma_0 u h) \psi_U(u) du.$$

Proof. Note that $U^{w_0} = \prod_{\alpha \in \Phi(U,T): w_0 \alpha > 0} U_\alpha$ while $U_0 = \prod_{\alpha \in \Phi(U,T): w_0 \alpha < 0} U_\alpha$. It follows that $U = U^{w_0} U_0$. One calculates that $\{\alpha \in \Phi(U, T) : w_0 \alpha > 0\}$ equals

$$\{(11110000); (11111000); (11121000); (11221000); (12232100); (12232110); (12232111)\}.$$

For each value of α , one easily checks that none of $\alpha - 00011100, \alpha - 00001110, \alpha - 00000111$ is a root. It follows that z normalizes U_0 , and hence $U = U^{w_0 z} U_0$.

Since $f_{s,\chi}(w_0 z u h)$ is left-invariant by $U^{w_0 z}(\mathbb{A})$, and since ψ_U is trivial on $U^{w_0 z}(\mathbb{A})$, the result follows. \square

Lemma 8. *The group $H \cap (w_0 z)^{-1} P_2 w_0 z$ is the diagonally embedded copy of G_2 , i.e., the subgroup of H generated by $x_{\pm 00010000}(r) x_{\pm 00000010}(r)$, and*

$$x_{\pm 01000000}(r) x_{\pm 00100000}(r) x_{\pm 00001000}(-r) x_{\pm 00000001}(r) x_{00111100}(r) x_{01011100}(-r).$$

Proof. The group $M_1 \cap w_{\text{lng}}^{-1} P_2 w_{\text{lng}}$ is the maximal parabolic subgroup of M_1 whose unipotent radical contains U_{α_3} . Denote this group by $P_3^{M_1}$. Then $H \cap (w_0 z)^{-1} P_2 w_0 z = H \cap (\nu_0 z)^{-1} P_3^{M_1} \nu_0 z = \{h \in H \mid \nu_0 z h z^{-1} \nu_0^{-1} \in P_3^{M_1}\}$. Now, M_1 is isogenous to SO_{14} . The kernel of this isogeny is contained in the maximal torus, and hence in every parabolic subgroup, and this means $\nu_0 z h z^{-1} \nu_0^{-1}$ lies in $P_3^{M_1}$ if and only if its image in SO_{14} lies in the corresponding parabolic subgroup of SO_{14} . This reduces the lemma to a straightforward matrix calculation. \square

This completes the proof of theorem 1. \square

3. THE UNRAMIFIED COMPUTATIONS

3.1. Notation. In this section the unramified local zeta integral corresponding to integral (4) will be computed. Therefore, let F now denote a nonarchimedean local field, and π an unramified representation of $G_2(F)$. In this section we write T_{E_8} for our fixed maximal torus of E_8 and use T for a fixed maximal torus of G_2 . Let f denote a section of the induced representation $\text{Ind}_{P_2(F)}^{E_8(F)} \delta_P^s$, where $s \in \mathbb{C}$ with $\text{Re}(s)$ large. (We no longer need to consider

characters of the type $\delta_P^s \chi$: if χ is unramified, then it is equal to δ_P^ν for some $\nu \in \mathbb{C}$ and we may absorb ν into s .) Let ω_π denote the normalized spherical function for π . Let $w_0 = w_{\text{lng}} \nu_0$ where

$$w_{\text{lng}} = w[24315423456542314354276542314354265437654287654231435426543765428765431]$$

and $\nu_0 = w[345678245673456]$. Let U denote the unipotent radical of the standard maximal parabolic subgroup of E_8 with Levi isomorphic to the product of $Spin_{14}$ and a one dimensional torus, and let U_0 be the T_{E_8} -stable subgroup of U such that $\Phi(U, T_{E_8}) \setminus \Phi(U_0, T_{E_8})$ equals

$$\{(11110000); (11111000); (11121000); (11221000); (12232100); (12232110); (12232111)\}.$$

This group can also be described as $U \cap w_0^{-1} \overline{U}_{\max} w_0$, or $\prod_{\alpha > 0: w_0 \alpha < 0} U_\alpha$. Define a character ψ_U of U (which may then be restricted to U_0) by the formula

$$\psi_U(u) = \psi_U(x_{11221111}(r_1)x_{11122111}(r_2)x_{12232210}(r_3)x_{11233210}(r_4)u') = \psi(r_1 + r_2 + r_3 + r_4),$$

for any $u' \in U$ which lies in the product of the groups U_α with $\alpha \in \Phi(U, T_{E_8})$ and $\alpha \notin \{11221111, 11122111, 12232210, 11233210\}$. Embed $G_2 \times G_2$ into E_8 as the identity component of the stabilizer of ψ_U in M_1 , as in section 2.2. Finally, let $z = x_{00011100}(1)x_{00001110}(1)x_{00000111}(-1)$.

Then the local zeta integral is given by

$$(16) \quad I(s, \pi) := \int_{G_2(F)} \int_{U_0(F)} \omega_\pi(g) f(w_0 z u(1, g), s) \psi_U(u) du dg$$

Throughout this section, we shall often abuse notation and denote the F -points of an algebraic group H by “ H ,” suppressing the ubiquitous “ (F) ” ’s. We denote the long and short simple roots of G_2 by β_{lng} and β_{sht} respectively.

We embed G_2 into SO_8 in such a way that

$$T = \{t = \text{diag}(t_1 t_2^2, t_1 t_2, t_2, 1, 1, t_2^{-1}, t_1^{-1} t_2^{-1}, t_1^{-1} t_2^{-2})\}.$$

This puts coordinates t_1, t_2 on T . Another way to define the same coordinates is that $t_1 = \beta_{\text{lng}}(t)$ and $t_2 = \beta_{\text{sht}}(t)$.

We also write $h : T \hookrightarrow E_8$ for the embedding $h(\beta_{\text{sht}}^\vee(a_1) \beta_{\text{lng}}^\vee(a_2)) = \alpha_2^\vee \alpha_3^\vee \alpha_5^\vee(a_1) \alpha_4^\vee(a_2)$ determined by our chosen embedding $G_2 \hookrightarrow E_8$. Composing with the projection $M_1 \rightarrow SO_{14}$ gives the embedding

$$T \hookrightarrow SO_{14}, \quad t \mapsto \begin{pmatrix} I_3 & & \\ & t & \\ & & I_3 \end{pmatrix},$$

which we also denote h . (Here T is identified with a subgroup of SO_8 .)

We also write K for $G_2(\mathfrak{o})$.

Remark 3. We saw in lemma 8 that the $H \cap (w_0 z)^{-1} P_2 w_0 z$ is the diagonally embedded copy of G_2 inside of $H = G_2 \times G_2$. Hence $\{(1, g) : g \in G_2\} \subset H$ maps isomorphically onto $(H \cap (w_0 z)^{-1} P_2 w_0 z) \setminus H$. In fact we could take the integral in (16) over any embedded copy of G_2 with this property.

3.2. Main Local Result.

Theorem 1. For $\text{Re}(s)$ sufficiently large,

$$I(s, \pi) = \frac{L(s, \pi, \text{St})}{N(s)},$$

where $L(s, \pi, \text{St})$ is the local L function attached to π and the 7 dimensional “standard” representation of $G_2(\mathbb{C})$, and $N(s)$ is the normalizing factor of the Eisenstein series, given explicitly (see section 4) by

$$N(s) = \zeta(17s) \prod_{i=2}^6 \zeta(17s - i) \prod_{i=5}^8 \zeta(34s - 2i) \zeta(51s - 21).$$

Proof. The proof occupies the rest of section 3, and comes in several steps. Write T for the maximal torus of G_2 , and write T^+ for the subset $\{t \in T : |\beta_{\text{lng}}(t)|, |\beta_{\text{sh}}(t)| \leq 1\}$. As a subset of SO_8 , this is

$$\{\text{diag}(t_1 t_2^2, t_1 t_2, t_2, 1, 1, t_2^{-1}, t_1^{-1} t_2^{-1}, t_1^{-1} t_2^{-2}) \in T : |t_1|, |t_2| \leq 1\}.$$

For the rest of the section, we define $t_1 := \beta_{\text{lng}}(t)$ and $t_2 := \beta_{\text{sh}}(t)$ as coordinates on the torus T . Define

$$\begin{aligned} \psi_{U,t}(x_{10111111}(r_1)x_{12232210}(r_2)x_{12232111}(r_3)x_{11233210}(r_4)x_{11232211}(r_5)x_{11222221}(r_6)u') = \\ \psi\left(\sum_{i=1}^4 r_i + t_2 r_5 + t_1 t_2 r_6\right), \end{aligned}$$

$$U'_0 := \nu_0 U_0 \nu_0^{-1} = U \cap w_0^{-1} \overline{U}_{\max} w_0, \text{ and}$$

$$(17) \quad I(s, t) := \int_{U'_0} f(w_0 u, s) \psi_{U,t}(u) du.$$

Then we prove in section 3.3 that

$$(18) \quad I(s, \pi) = \int_{T^+} \omega_\pi(t) \delta(t) |t_1 t_2^2|^{17s-5} I(s, t) dt,$$

where $\delta(t)$ is the measure of the double coset KtK .

It's value is given by $\delta_B(t)^{-1} Q/Q_t$, where

$$Q = \frac{(1 - q^{-2})(1 - q^{-6})}{(1 - q^{-1})^2} = (1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + 2q^{-4} + 2q^{-5} + q^{-6}),$$

$$Q_t := \begin{cases} Q, & |t_1| = |t_2| = 1, \\ 1 + q^{-1}, & 0 = \max(|t_1|, |t_2|) > \min(|t_1|, |t_2|), \\ 1 & 0 > \max(|t_1|, |t_2|). \end{cases}$$

This is proved in section 3.2 of [M]. See also [Cass].

Every term in the integrand of (18) depends only on the \mathfrak{p} -adic valuations of t_1 and t_2 of T . To be precise, suppose that $v(t_2) = m$ and $v(t_1) = n$. Here v is the \mathfrak{p} -adic valuation. Then $\delta_B(t) = q^{-6n-10m}$. Define $Q_{(m,n)} = Q_t$ for $t \in T$ with $t_1 = p^n$ and $t_2 = p^m$.

Now, set $x = q^{-17s}$, and define

$$Z(x, q) = (1-x)(1-xq^2)(1-xq^3)(1-xq^4)(1-x^2q^{10})(1-x^2q^{12})$$

and

$$I_0(n, m; x, q) := (1-xq^6)(1-x^3q^{21}) - (1-xq^5)(1-xq^6)(xq^8)^{m+1} - (1-xq^8)(1-xq^5)(xq^8)^m(xq^7)^{n+1}.$$

Then theorem 3 states that

$$I(s, t) = \frac{Z(x, q)I_0(n, m; x, q)}{(1-xq^7)(1-xq^8)}.$$

Moreover, $|t_1 t_2^{17s-5}| = (xq^5)^{2m+n}$. This leaves $\omega_\pi(t)$. To write a formula for $\omega_\pi(t)$, it is convenient to identify the pair (m, n) with the weight $m\varpi_1 + n\varpi_2$ in the weight lattice Λ of ${}^L G_2^0 = G_2(\mathbb{C})$. This is compatible with the natural identification of Λ with $T/T(\mathfrak{o})$ which is built into the definition of the L -group. Let τ be an element of the L -group associated to the representation π (this determines τ up to the action of W which is enough). We write λ additively, and therefore denote the value of $\lambda \in \Lambda$ at τ by τ^λ as opposed to $\lambda(\tau)$. Define S_0 to be the finite set of weights of ${}^L G^0$ which have at least one expression as a sum of distinct positive roots, and define polynomials P_ν , ($\nu \in S_0$) by the condition

$$\left(\prod_{\alpha > 0} 1 - q^{-1} \tau^{-\alpha} \right) = \sum_{\nu \in S_0} P_\nu(q^{-1}) \tau^{-\nu}.$$

Then

$$\omega_\pi(t) = \frac{1}{Q} q^{-5m-3n} \sum_{\nu \in S_0} P_\nu(q^{-1}) \frac{A_{\varpi+\rho-\nu}(\tau)}{A_\rho(\tau)}, \quad \text{where} \quad A_\lambda(\tau) = \sum_{w \in W} (-1)^{\ell(w)} \tau^{w\lambda}, \quad (\lambda \in \Lambda).$$

Here ℓ denotes the length function on W and Q is defined as in the formula for the measure of KtK given above. It is not difficult to derive this expression for $\omega_\pi(t)$ from the one given in [Ca].

Plugging all of this into 18 yields

$$I(s, \pi) = \frac{Z(x, q)}{(1-xq^7)(1-xq^8)} \sum_{m,n=0}^{\infty} \frac{x^{2m+n} q^{15m+8n} I_0(m, n; x, q)}{Q_{(m,n)}} \sum_{\nu \in S_0} P_\nu(q^{-1}) \frac{A_{(m,n)-\nu+\rho}(\tau)}{A_\rho(\tau)}.$$

The inner summation must be calculated. This is accomplished in theorem 2, which states that

$$(19) \quad \sum_{m,n=0}^{\infty} \frac{x^{2m+n} q^{15m+8n} I_0(m, n; x, q)}{Q_{(m,n)}} \sum_{\nu \in S_0} P_{\nu}(q^{-1}) \frac{A_{(m,n)-\nu+\rho}(\tau)}{A_{\rho}(\tau)} \\ = (1 - xq^5)(1 - xq^6)(1 - xq^7)(1 - xq^8)(1 - x^2q^{14})(1 - x^3q^{21}) \sum_{r=0}^{\infty} \chi_{(r,0)}(\tau) x^r q^{8r},$$

where χ_{λ} is the character of the irreducible finite dimensional representation of $G_2(\mathbb{C})$ having highest weight λ . The decomposition of the symmetric algebra of St is described by the theorem of Brion [Br], and this description implies that

$$L(s, \pi, \text{St}) = (1 - x^2 q^{16})^{-1} \sum_{r=0}^{\infty} \chi_{(r,0)}(\tau) x^r q^{8r}.$$

Plugging this in, and noting that $(1 - xq^5)(1 - xq^6)(1 - x^2q^{14})(1 - x^2q^{16})(1 - x^3q^{21})Z(x, q)$ is precisely $N(s)$ yields theorem 1. \square

3.3. Transformation of the integral.

Proposition 4. *Let $I(s, \pi)$ be defined by (16), and $I(s, t)$ by (17). Then*

$$I(s, \pi) = \int_{T^+} \omega_{\pi}(t) \delta(t) |t_1 t_2^2|^{17s-5} I(s, t) dt.$$

Proof. Using the bi-K invariant property of ω_{π} , integral (16) is equal to

$$(20) \quad \int_{T^+} \int_{U_0} \omega_{\pi}(t) f(w_0 z u h(t), s) \psi_U(u) \delta(t) du dt,$$

where $\delta(t)$ denotes the measure of the double coset KtK .

Conjugate $h(t)$ across u . It normalizes U_0 preserving ψ_U , but the change of variables $u \mapsto h(t)uh(t)^{-1}$ produces a factor of

$$\left| t^{\sum_{\alpha \in \Phi(U_0, T_{E_8})} \alpha} \right| = |t_1 t_2^2|^{-5}.$$

from the measure. (Here $t_1 = \beta_{\text{lng}}(t)$ and $t_2 = \beta_{\text{sht}}(t)$.)

Next define $z(t) = h(t)^{-1} z h(t)$. Then $z(t) = x_{00011100}(t_2) x_{00001110}(t_1 t_2) x_{00000111}(t_1 t_2^2)$. This calculation can be done in LiE or with matrices, since the projection $M_1 \rightarrow SO_{14}$ restricts to an isomorphism on $G_2 \cdot U$. The image of t was described above, and the image of z is the unipotent matrix $I_{14} + e'_{1,4} + e'_{2,5} + e'_{3,6}$ in SO_{14} . Now conjugate z across u_0 to obtain that integral (20) is equal to

$$(21) \quad \int_{T^+} \int_{U_0} \omega_{\pi}(t) f(w_0 u h(t) z(t), s) [z(t) \cdot \psi_U](u) \delta(t) du dt$$

Notice that $z(t)$ is in the maximal compact and hence can be ignored.

Write $w_0 = w_{\text{lng}}\nu_0$ as in Theorem 1. We conjugate $h(t)$ to the left and then ν_0 to the right, and we obtain

$$(22) \quad \int_{T^+} \int_{U'_0} \omega_\pi(t) f(w_{\text{lng}}u, s) [\nu_0 z(t) \cdot \psi_U](u) \delta(t) |t_1 t_2^2|^{-5} \delta_{P_2}^s(w_0 h(t) w_0^{-1}) du dt.$$

The character $\nu_0 z(t) \cdot \psi_U(u)$ is precisely $\psi_{U,t}$, and $\delta_{P_2}^s(w_0 h(t) w_0^{-1}) = |t_1 t_2^2|^{17s}$. This gives the result. \square

We remark that U'_0 may be defined as $\nu_0 U_0 \nu_0^{-1}$, as $U \cap w_0^{-1} \overline{U}_{\max}$, or as the T_{E_8} -stable subgroup of U such that $\Phi(U, T_{E_8}) \setminus \Phi(U'_0, T_{E_8})$ is the set

$$\{(12343210); (12343211); (12343221); (12343321); (12344321); (12354321); (13354321)\}$$

Thus $\dim U = 71$.

3.4. Proof of Main Unramified Identity. In this section we prove (19). Set

$$\begin{aligned} I_0(n, m; x, q) &= 1 - xq^6 - x^3q^{21} + x^4q^{27} - x^{m+1}q^{8(m+1)} + x^{m+2}q^{8m+14} - x^{n+m+1}q^{7(n+1)+8m} \\ &\quad + x^{n+m+2}q^{7n+8m+12} + x^{n+m+2}q^{7n+8m+15} - x^{n+m+3}q^{7n+8m+20} + x^{m+2}q^{8m+13} - x^{m+3}q^{8m+19} \\ &= (1 - xq^6)(1 - x^3q^{21}) - (xq^8)^{m+1}(1 - xq^5)(1 - xq^6) - (xq^7)^{n+1}(xq^8)^m(1 - xq^5)(1 - xq^8) \end{aligned}$$

Identify \mathbb{Z}^2 with the weight lattice of $G_2(\mathbb{C})$ via the mapping $(n, m) \mapsto n\varpi_1 + m\varpi_2$, and regard I_0 as a function defined on the weight lattice as well.

Write (n, m) for the trace of the semisimple conjugacy class in $G_2(\mathbb{C})$ attached to π , acting on the irreducible finite dimensional representation of $G_2(\mathbb{C})$ with highest weight $n\varpi_1 + m\varpi_2$. Here ϖ_1 and ϖ_2 are the fundamental weights.

Define S_0 and P_ν as in the previous section and set

$$Q_\varpi = \begin{cases} Q, & \varpi = 0 \\ 1 + q^{-1}, & \varpi = n\varpi_1 \text{ or } n\varpi_2, \\ 1, & \text{otherwise,} \end{cases}$$

for ϖ a dominant weight of $G_2(\mathbb{C})$.

Define

$$p(\varpi) = \frac{1}{Q_\varpi} \sum_{\nu \in S_0} P_\nu(q^{-1}) \frac{\varpi + \rho - \nu}{|\varpi + \rho - \nu|} \chi_{|\varpi + \rho - \nu|}(\tau),$$

and $p(n, m) = p(n\varpi_1 + m\varpi_2)$.

Define

$$z_0(x, q) = (1 - xq^5)(1 - xq^6)(1 - xq^7)(1 - xq^8)(1 - x^2q^{14})(1 - x^3q^{21}).$$

Theorem 2. *We have*

$$(23) \quad \sum_{n,m=0}^{\infty} p(n, m) I_0(n, m; x, q) x^{n+2m} q^{8n+15m} = z_0(x, q) \sum_{r=0}^{\infty} (r, 0) x^r q^{8r}$$

Proof. For ϖ a dominant weight of $G_2(\mathbb{C})$, define $J(\varpi; x, q) = I_0(\varpi; x, q) \tau_0^{\varpi} / Q_{\varpi}$, where τ_0 is an element of the standard maximal torus ${}^L T$ of $G_2(\mathbb{C})$, chosen so that $\tau_0^{\varpi_1} = xq^8$ and $\tau_0^{\varpi_2} = x^2q^{15}$. (Here, we employ the exponential notation for the weights. That is ϖ is a function from ${}^L T$ to \mathbb{C}^{\times} , and its value at $\tau \in {}^L T$ is denoted τ^{ϖ} .) Then

$$J(\varpi; x, q) = \frac{1}{Q_{\varpi}} \left((1 - xq^6)(1 - x^3q^{21})\tau_0^{\varpi} - (xq^8)(1 - xq^5)(1 - xq^6)\tau_1^{\varpi} - (xq^7)(1 - xq^5)(1 - xq^8)\tau_2^{\varpi} \right),$$

where $\tau_1, \tau_2 \in {}^L T$ satisfy

$$\tau_1^{\varpi_1} = xq^8, \quad \tau_1^{\varpi_2} = x^3q^{23}, \quad \tau_2^{\varpi_1} = x^2q^{15}, \quad \tau_2^{\varpi_2} = x^3q^{23}.$$

Set

$$J_0(x, q) = (1 - xq^6)(1 - x^3q^{21}), \quad J_1(x, q) = (xq^8)(1 - xq^5)(1 - xq^6), \quad J_2(x, q) = (xq^7)(1 - xq^5)(1 - xq^8),$$

so that

$$J(\varpi; x, q) = J_0(x, q)\tau_0^{\varpi} - J_1(x, q)\tau_1^{\varpi} - J_2(x, q)\tau_2^{\varpi}.$$

Write Λ^{++} for the semigroup of dominant weights of $G_2(\mathbb{C})$. For $\lambda \in \Lambda^{++}$, write $\chi(\lambda)$ for the character of the irreducible finite dimensional representation of $G_2(\mathbb{C})$ with highest weight λ .

Then $\chi(\lambda)$ appears in the expression for $p(\varpi)$ given in the previous section if and only if $\{(w, \nu) \mid w \in W, \nu \in S_0, \varpi + \nu + \rho = w(\lambda + \rho)\}$ is nonempty. If this set is nonempty, then the coefficient of $\chi(\lambda)$ in this expression for $p(\varpi)$ is governed by

$$D(\lambda, \varpi) := \{(w, S) \mid w \in W, \nu \in S_0, S \subset \Phi^+, \varpi + \Sigma(S) + \rho = w(\lambda + \rho)\},$$

where $\Sigma(S) := \sum_{\alpha \in S} \alpha$. To be precise

$$p(\varpi) = \sum_{\lambda \in \Lambda^{++}} \sum_{(w, S) \in D(\lambda, \varpi)} (-1)^{\ell(w)} (-q^{-1})^{|S|}$$

Thus, it must be shown that for $\lambda \in \Lambda^{++}$

$$\sum_{\varpi \in \Lambda^{++}} \sum_{(w, S) \in D(\lambda, \varpi)} J(\varpi; x, q) (-1)^{\ell(w)} (-q^{-1})^{|S|} = \begin{cases} z_0(x, q) (xq^8)^r, & \varpi = r\varpi_1, \\ 0, & \text{otherwise.} \end{cases}$$

This will be reduced to a finite number of cases.

Lemma 9. *Given $\lambda, \varpi \in \Lambda^{++}$, $D(\lambda, \varpi)$ is nonempty if and only if $\varpi \in \lambda + S_0$.*

Proof. The set $\rho - S_0 = \{\frac{1}{2}\Sigma(S) - \frac{1}{2}\Sigma(S^c) \mid S \subset \Phi\}$ is clearly W -stable. (Here S^c denotes the complement of S in Φ .) So

$$\exists S \subset \Phi \text{ s.t. } w(\varphi - \Sigma(S) + \rho) = \lambda + \rho \iff \exists S \subset \Phi \text{ s.t. } w\varphi - \Sigma(S) + \rho = \lambda + \rho \iff w\varphi \in \lambda + S_0.$$

Now, the convex hull of $\lambda + S_0$ is a convex dodecagon D with edges parallel to the roots. The center of mass of the dodecagon D is $\lambda + \rho$ and lies in the positive Weyl chamber. Let ℓ be any line which is parallel to a root and does not pass through $\lambda + \rho$. Then ℓ partitions the plane into two half-planes. Let H^+ be the half-plane containing $\lambda + \rho$ and H^- the other. Then the reflection of $(D \cap H^+)$ over ℓ contains H^- . Applying this geometric observation to the reflections in the Weyl group, it is clear that $\{w \in W : w\varphi \in \lambda + S_0\}$ must contain the identity if it is nonempty. (In fact, one can say more. If $w\varphi \in \lambda + S_0$ and $w = w[i_1 i_2 \dots i_k]$ is a reduced expression for w , then $w[i_j \dots i_k]\varphi \in \lambda + S_0$ for $1 \leq j \leq k$.) \square

Thus we need to show that

(24)

$$\sum_{\substack{\nu \in S_0 \\ \lambda + \nu \in \Lambda^{++}}} J(\lambda + \nu; x, q) \sum_{\substack{w \in W \\ w(\lambda + \rho) \in \lambda + \nu + \rho - S_0}} (-1)^{\ell(w)} \sum_{\substack{S \subset \Phi \\ \lambda + \nu + \rho - \Sigma(S) = w(\lambda + \rho)}} (-q^{-1})^{|S|} = \begin{cases} z_0(x, q)(xq^8)^r, & \lambda = r\varpi_1, \\ 0, & \text{otherwise.} \end{cases}$$

Now, $\{w \mid w(\lambda + \rho) \in \varphi + \rho - S_0\}$ for some $\varphi \in \Lambda^{++}$ is given by

$$\begin{cases} \{e\}, & \lambda = n\varpi_1 + m\varpi_2, \ n \geq 5, m \geq 3, \\ \{e, w[1]\}, & \lambda = n\varpi_1 + m\varpi_2, \ n \leq 4, m \geq 3, \\ \{e, w[2]\}, & \lambda = n\varpi_1 + m\varpi_2, \ n \geq 5, m \leq 2. \end{cases}$$

Here e is the identity element of the Weyl group.

Lemma 10. Fix n with $0 \leq n \leq 4$ and $\nu \in S_0$. For $\lambda = n\varpi_1 + m\varpi_2$, the expression

$$P(n, \nu; q^{-1}) := \sum_{\substack{w \in W \\ w(\lambda + \rho) \in \lambda + \nu + \rho - S_0}} (-1)^{\ell(w)} \sum_{\substack{S \subset \Phi \\ \lambda + \nu + \rho - \Sigma(S) = w(\lambda + \rho)}} (-q^{-1})^{|S|}$$

is independent of $m \geq 3$. Likewise, for fixed $m \leq 2$ it is independent of $n \geq 5$.

Proof. We prove the first statement. The proof of the second statement is symmetric. Indeed, for $m \geq 3$ and $n \leq 4$, the given expression is

$$\sum_{\substack{S \subset \Phi \\ \nu = \Sigma(S)}} (-q^{-1})^{|S|} + \sum_{\substack{S \subset \Phi \\ \lambda + \nu + \rho - \Sigma(S) = w[1](\lambda + \rho)}} (-q^{-1})^{|S|},$$

and $w[1]\lambda - \lambda = n\alpha_1$, independent of m . \square

Observe that for $n \leq 4$ and $m \geq 3$, $\{\nu \in S_0 \mid \lambda + \nu \in \Lambda^{++}\}$ is also independent of m . Thus, for all $m \geq 3$, the left hand side of equation (24) is equal to

$$\begin{aligned}
& \sum_{\substack{\nu \in S_0 \\ \nu = a\varpi_1 + b\varpi_2, a \geq -n}} J(\lambda + \nu; x, q) P(n, \nu; q^{-1}) \\
&= \sum_{\substack{\nu \in S_0 \\ \nu = a\varpi_1 + b\varpi_2, a \geq -n}} (J_0(x, q)\tau_0^{\lambda+\nu} - J_1(x, q)\tau_1^{\lambda+\nu} - J_2(x, q)\tau_2^{\lambda+\nu}) P(n, \nu; q^{-1}) \\
&= (x^2 q^{15})^m \left((xq^8)^n J_0(x, q) \sum_{\nu} P(n, \nu; q^{-1}) \tau_0^{\nu} \right) \\
&\quad - (x^3 q^{23})^m \left((xq^8)^n J_1(x, q) \sum_{\nu} P(n, \nu; q^{-1}) \tau_1^{\nu} + (x^2 q^{15})^n J_2(x, q) \sum_{\nu} \tau_2^{\nu} P(n, \nu; q^{-1}) \right).
\end{aligned}$$

Now $m \mapsto (x^2 q^{15})^m$ and $m \mapsto (x^3 q^{23})^m$ are two linearly independent functions of m . Therefore, after checking equation 24 for two distinct values of $m \geq 3$, one may deduce that

$$\sum_{\nu} P(n, \nu; q^{-1}) \tau_0^{\nu} = \left((xq^8)^n J_1(x, q) \sum_{\nu} P(n, \nu; q^{-1}) \tau_1^{\nu} + (x^2 q^{15})^n J_2(x, q) \sum_{\nu} \tau_2^{\nu} P(n, \nu; q^{-1}) \right) = 0,$$

and thence that (24) holds for all $m \geq 3$. The same method allows one to reduce the case when $m \leq 2$ is fixed and $n \geq 5$ is arbitrary to checking two cases, and to reduce the case $m \geq 3, n \geq 5$ to checking three cases. Overall, it suffices to check all pairs (n, m) with $n \leq 6$ and $m \leq 4$. This is easily accomplished using LiE [L]. \square

Remark 4. Equation (24) has another simple proof in the case $n \geq 5, m \geq 3$. In that case, equation (24) reduces to

$$\sum_{\nu \in S_0} J(\lambda + \nu; x, q) \sum_{\substack{S \subset \Phi^+ \\ \Sigma(S) = \nu}} (-q^{-1})^{|S|} = 0.$$

The left hand side is equal to

$$\begin{aligned}
& \sum_{S \subset \Phi^+} (-q^{-1})^{|S|} J(\lambda + \Sigma(S); x, q) \\
&= J_0(x, q) \prod_{\alpha \in \Phi^+} (1 - q^{-1} \tau_0^{\alpha}) \tau_0^{\lambda} - J_1(x, q) \prod_{\alpha \in \Phi^+} (1 - q^{-1} \tau_1^{\alpha}) \tau_1^{\lambda} - J_2(x, q) \prod_{\alpha \in \Phi^+} (1 - q^{-1} \tau_2^{\alpha}) \tau_2^{\lambda}.
\end{aligned}$$

But one can fairly easily check that for each $i = 0, 1, 2$, there is a positive root α such that $\tau_i^{\alpha} = q$.

4. THE NORMALIZING FACTOR OF THE EISENSTEIN SERIES

In this section we compute the normalizing factor of the Eisenstein series. The Eisenstein series we use, denoted by $E(h, s)$, is attached to the induced representation $\text{Ind}_{P_2(\mathbb{A})}^{E_8(\mathbb{A})} \delta_{P_2}^s \chi$. Here, $\chi = \prod_v \chi_v$ is a character of $\mathbb{A}^{\times}/F^{\times}$ which has been identified with a character of

$P_2(F) \backslash P_2(\mathbb{A})$ by composing it with the rational character $\det^{\frac{1}{2}}$ of P_2 . (See section 2.1.) To compute the normalizing factor we consider an unramified place v , where $\chi_v = \delta_{P_2}^\nu$ for $\nu \in \mathbb{C}$. Then $\text{Ind}_{P(F_v)}^{E_8(F_v)} \delta_{P_2}^s \chi$ is a sub-representation of $\text{Ind}_{B(F_v)}^{E_8(F_v)} (\delta_{P_2}^{s+\nu} \delta_B^{-1/2}) \delta_B^{1/2}$, where B is the Borel of E_8 . Any root $\alpha = \sum n_i \alpha_i$ such that $n_2 > 0$ will contribute the factor

$$\frac{\zeta_v(17n_2(s+\nu) - \sum n_i)}{\zeta_v(17n_2(s+\nu) - \sum n_i + 1)} = \frac{L_v(17n_2s - \sum n_i, \chi_v^{n_2})}{L_v(17n_2s - \sum n_i + 1, \chi_v^{n_2})}.$$

See [PS-R], proposition 5.2. Thus, after cancellation we obtain the factor the product $Z_1(s)Z_2(s)$ where

$$Z_1(s) = \frac{L(17s-10, \chi)}{L(17s, \chi)} \frac{L(17s-11, \chi)}{L(17s-2, \chi)} \frac{L(17s-12, \chi)}{L(17s-3, \chi)} \frac{L(17s-13, \chi)}{L(17s-4, \chi)} \frac{L(17s-14, \chi)}{L(17s-5, \chi)} \frac{L(17s-16, \chi)}{L(17s-6, \chi)}$$

$$Z_2(s) = \frac{L(34s-17, \chi^2)}{L(34s-10, \chi^2)} \frac{L(34s-19, \chi^2)}{L(34s-12, \chi^2)} \frac{L(34s-21, \chi^2)}{L(34s-14, \chi^2)} \frac{L(34s-23, \chi^2)}{L(34s-16, \chi^2)} \frac{L(51s-29, \chi^3)}{L(51s-21, \chi^3)}$$

The denominator is the normalizing factor, and for $\text{Re}(s) > \frac{1}{2}$ the poles of the Eisenstein series should be determined by the poles of the numerator. Thus we expect the following to hold.

Conjecture 1. *For $\text{Re}(s) > 1/2$, the possible poles of the Eisenstein series are as follows.*

- If χ is trivial, then the Eisenstein series $E(h, s)$ can have a double pole at the points $\frac{10}{17}$, $\frac{11}{17}$ and $\frac{12}{17}$. At the points $\frac{9}{17}$; $\frac{13}{17}$; $\frac{14}{17}$; $\frac{15}{17}$ and 1, it can have a simple pole.
- If χ is nontrivial quadratic, then the Eisenstein series can have simple poles at $\frac{10}{17}$, $\frac{11}{17}$, $\frac{12}{17}$, $\frac{13}{17}$, $\frac{14}{17}$, $\frac{15}{17}$ and 1.
- If χ is nontrivial cubic, then the Eisenstein series can have a simple pole at $\frac{10}{17}$.
- If the order of χ exceeds 3 then the Eisenstein series is holomorphic in $\text{Re}(s) > \frac{1}{2}$.

5. CALCULATION OF $I(s, t)$

Theorem 3. *Let $I(s, t)$ be defined by (17), let*

$$Z(x, q) = (1-x)(1-xq^2)(1-xq^3)(1-xq^4)(1-x^2q^{10})(1-x^2q^{12})$$

and

$$I_0(n, m; x, q) := (1-xq^6)(1-x^3q^{21}) - (1-xq^5)(1-xq^6)(xq^8)^{m+1} - (1-xq^8)(1-xq^5)(xq^8)^m(xq^7)^{n+1}.$$

Then

$$I(s, t) = \frac{Z(x, q)I_0(n, m; x, q)}{(1-xq^7)(1-xq^8)},$$

where $x = q^{-17s}$, n is the \mathfrak{p} -adic valuation of t_1 , and m is that of t_2 .

5.1. First reduction. The purpose of this section is to compute the period $I(s, t)$ appearing in (18). The first step is to reduce the study of $I(s, t)$ to the study of a simpler period $J(a, b, c)$, which we now define.

Throughout section 5, we denote the maximal torus of E_8 by T . If w is an element of the Weyl group, W , of E_8 , then we define $U_w = U_{\max} \cap w^{-1}\overline{U}_{\max}w = \prod_{\alpha>0: w\alpha<0} U_\alpha$. In this notation, the group U'_0 appearing in the definition of $I(s, t)$ can also be described as $U_{w_{\text{ing}}}$.

Definition 1. For $a, b, c \in F^\times$, define

$$(25) \quad J(a, b, c) = \int_{U_w} f_s(wu) \psi(u_{00100000} + au_{01000000} + u_{10000000} + bu_{00001110} + cu_{00000111}) du, \\ w = w[243154234565423145765423187].$$

Theorem 4.

$$I(s, t) = \begin{cases} (1 + q^{-51s+18})J(1, 1, 1), & t_1, t_2 \in \mathfrak{o}^\times, \\ J(1, 1, t_1) - q^{-68s+26}J(1, 1, p^{-2}t_1), & t_2 \in \mathfrak{o}^\times, t_1 \notin \mathfrak{o}^\times, \\ J(1, t_2, t_1t_2) - q^{-34s+14}J(p, p^{-1}t_2, p^{-1}t_1, t_2) + q^{-81s+35}J(1, p^{-2}t_2, p^{-2}t_1t_2), & t_2 \notin \mathfrak{o}^\times. \end{cases}$$

5.1.1. *Tools.* Before proceeding to the details of the proof we review a few standard techniques which are used in the calculations. We consider integrals of the following type:

$$(26) \quad \int_V f_s(wu) \psi_V(u) du,$$

where $w \in W$,

$$V \subset \{u = x_{\beta_1}(u_1) \dots, x_{\beta_N}(u_N) : u_i \in F, (1 \leq i \leq N)\},$$

is defined by a finite set of conditions of one of the following three types:

$$|u_i| \leq 1, \quad |u_i| > 1, \quad u_i = c, \quad c \in F, \quad \text{constant}.$$

Also β_1, \dots, β_N are distinct roots such that $w\beta_i \notin \Phi(P, T)$, $(1 \leq i \leq N)$, and

$$\psi_V(x_{\beta_1}(u_1) \dots, x_{\beta_N}(u_N)) = \psi \left(\sum_{i=1}^N c_i u_i \right), \quad \text{for some } c_1, \dots, c_N \in F.$$

The variable u_i is said to be *free* if it does not appear in any of the conditions which define V . Of course, $I(s, t)$ is of this type.

The basic technique is to split an integral of this type up according to whether $|u_N| \leq 1$ or $|u_N| > 1$ and plug in the Iwasawa decomposition for each to obtain two integrals of the same type with a smaller value of N . Some care must be taken with regard to the order of the terms in the product to avoid venturing outside of this relatively simple class of integrals.

In addition to this basic technique, there are a few additional tricks that can be used.

Indeed, it's clear that the terms may be reordered arbitrarily if we assume that

$$(R) \quad \beta_i + \beta_j \text{ is a root} \implies \begin{cases} \beta_i + \beta_j = \beta_k \text{ for some } 1 \leq k \leq N, \\ c_k = 0, \\ u_k \text{ is free.} \end{cases}$$

Lemma 11. *Suppose that u_N is not constant, and $c_N \notin \mathfrak{o}$. Then the section integral (26) is zero.*

Proof. Introduce $x_{\beta_N}(z)$ at right with $z \in \mathfrak{o}$ such that $\psi(c_N z) \neq 1$, and make a change of variables in u_N . \square

Lemma 12 (Killing). *Let*

$$(27) \quad V' = \{x_{\beta_1}(u_1) \dots, x_{\beta_{N-1}}(u_{N-1})\}, \quad \psi_{V'}(x_{\beta_1}(u_1) \dots x_{\beta_{N-1}}(u_{N-1})) = \psi \left(\sum_{i=1}^{N-1} c_i u_i \right).$$

Assume that

$$\int_{V'} f(wv' x_{\beta_N}(u_N) x_{\alpha}(z)) \psi_{V'}(v') dv = \psi(az + \varepsilon u_N z) \int_{V'} f(wv' x_{\beta_N}(u_N)) \cdot \psi_{V'}(v') dv,$$

*where $a \in \mathfrak{o}$ and $\varepsilon \in \mathfrak{o}^\times$. Then one may restrict u_N to \mathfrak{o} without affecting the value of the integral. In this situation we say we can **kill** the root β_N .*

Proof. Simply integrate z over \mathfrak{o} . \square

Lemma 13. *Suppose that $c_N \in \mathfrak{o}^\times$, and that there is a cocharacter $h_0 : GL_1 \rightarrow T$ with the property that $\langle h_0, \beta_N \rangle = 1$, and $\langle h_0, \beta_i \rangle = 0$ for all $i \neq N$ with $c_i \neq 0$, and that the variable u_N in (26) is subject to the bound $|u_N| > 1$. Then the section integral (26) is equal to the integral*

$$- \int_{V'} f_s(wu x_{\beta_N}(p^{-1})) \psi_{V'}(u) du.$$

Here, V' and $\psi_{V'}$ are defined as in 12.

Proof. Introduce $h_0(\varepsilon)$ at right and integrate ε over \mathfrak{o}^\times . Conjugating $h_0(\varepsilon)$ across and making suitable changes of variable, one obtains an inner integration of

$$q^k \int_{|\varepsilon|=1} \psi(c_N \varepsilon p^{-k}) d\varepsilon = \begin{cases} -1, & k = 1 \\ 0 & \text{otherwise,} \end{cases}$$

which gives the result. \square

Remark 5. *A sufficient condition for the existence of a cocharacter with the given properties is that $\#\{i : c_i \neq 0\} \leq 8$, and that there is an element of $SL(8, \mathbb{Z})$ with the property that each root β_i with $c_i \neq 0$ is one of the rows.*

Lemma 14. *Keep the notation and assumptions of lemma 12, but now assume that*

$$\int_{V'} f(wv'x_{\beta_N}(u_N)x_\alpha(z))\psi_{V'}(v') dv = \psi((a + bu_k + \varepsilon u_N)z) \int_{V'} f(wv'x_{\beta_N}(u_N) \cdot \psi_{V'}(v')) dv,$$

for some $a, b \in \mathfrak{o}, \varepsilon \in \mathfrak{o}^\times$, and $k \in \{1, \dots, N-1\}$. Then the section integral (26) is equal to

$$\int_{V'} f_s(wu)\psi\left(\sum_{i=1}^{N-1} c_i u_i + c_N b u_k\right) du,$$

where V' is the subset of V defined by the condition $u_N = -\varepsilon^{-1} b u_k$.

Proof. The proof is the same as that of lemma 12. □

Lemma 15. *Suppose that $w = w_1 w_2$. Then one may conjugate w_2 from right to left without changing the value of the integral.*

Remark 6. *In general $w_2 x_{\beta_i}(u_i) w_2^{-1} = x_{w_2 \beta}(\pm u_i)$. Normally, we make changes of variables at the same time to remove this signs. This may introduce signs into ψ_V .*

If S is any subset of $\Phi(G, T)$ which is closed under addition, then the product of the groups U_α , $\alpha \in S$, is a group, denoted U_S . Note that $S = \Phi(U_S, T)$. It will frequently be convenient to describe a subgroup U_S of U_w by specifying the complement of $\Phi(U_S, T)$ in $\Phi(U_w, T)$.

Proof of first reduction theorem 4. Using lemma 12, one kills

$$(28) \quad S = \left\{ \begin{array}{l} 10100000, 10110000, 10111000, 10111100, 10111110, \\ 11110000, 11111000, 11111100, 11121000, 11221000 \end{array} \right\},$$

deducing that $I(s, t)$, is equal to an integral of the same type, a 61-dimensional subgroup $U' \subset U_0$. let S'' be the complement of $\{11111110, 11121100, 10111111, 10000000\}$ in S' . Number the elements of S'' , as $\beta_1, \dots, \beta_{57}$, and let

$$U'' = \{x_{\beta_1}(u_{\beta_1}) \dots, x_{\beta_{57}}(u_{\beta_{57}}) : u_i \in F, (1 \leq i \leq 57)\}.$$

Define

$$II(s, t; m_1, m_2, m_3, n) = \int_{U''} f_s(w_{\text{ing}} u x_{11111110}(m_2) x_{11121100}(m_3) x_{10111111}(m_1) x_{10000000}(n)) \psi_{U, t}(u) du.$$

Thus

$$I'(s, t) = \int_F \int_F \int_F \int_F II(s, t; m_1, m_2, m_3, n) \psi(m_1) dm_2 dm_3 dm_1 dn.$$

We then consider various cases, based on the absolute values of n, m_1, t_1 , and t_2 . Write $I_{\mathfrak{o}, \mathfrak{o}}(s, t)$ for the integral over $n \in \mathfrak{o}$ and $m_1 \in \mathfrak{o}$, $I_{\mathfrak{o}, F \setminus \mathfrak{o}}(s, t)$ for the integral over $n \in \mathfrak{o}$ and $m_1 \in F \setminus \mathfrak{o}$, and so on, and $I_{F \setminus \mathfrak{o}}$ for the integral over $n \in F \setminus \mathfrak{o}$.

Using lemma 12 then lemma 15, yields $I_{\mathfrak{o}, \mathfrak{o}} = J(1, t_2, t_1 t_2)$.

Proposition 5. *One has*

$$I_{F \setminus \mathfrak{o}}(s, t) = \begin{cases} q^{-51s+18} J(1, 1, 1), & t_1, t_2 \in \mathfrak{o}^\times, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The four terms in

$$x_{111111110}(m_2)x_{11121100}(m_3)x_{101111111}(m_1)x_{100000000}(n)$$

all commute with one another. We rearrange them as

$$x_{111111110}(m_2)x_{100000000}(n)x_{11121100}(m_3)x_{101111111}(m_1).$$

And write $I'_{F \setminus \mathfrak{o}}(s, t) = I'_{F \setminus \mathfrak{o}, \mathfrak{o}}(s, t) + I'_{F \setminus \mathfrak{o}, F \setminus \mathfrak{o}}(s, t)$, where

$$\begin{aligned} I'_{F \setminus \mathfrak{o}, \mathfrak{o}}(s, t) &:= \int_{F \setminus \mathfrak{o}} \int_{\mathfrak{o}} \int_F \int_F II(s, t; m_1, m_2, m_3, n) \psi(m_1) dm_2 dm_3 dm_1 dn \\ &= \int_{F \setminus \mathfrak{o}} \int_F \int_F II(s, t; 0, m_2, m_3, n) dm_2 dm_3 dn \\ I'_{F \setminus \mathfrak{o}, F \setminus \mathfrak{o}}(s, t) &:= \int_{F \setminus \mathfrak{o}} \int_{F \setminus \mathfrak{o}} \int_F \int_F II(s, t; m_1, m_2, m_3, n) \psi(m_1) dm_2 dm_3 dm_1 dn. \end{aligned}$$

In $I'_{F \setminus \mathfrak{o}, \mathfrak{o}}(s, t)$, the root 01111110 kills 11121100, and then, plugging in the Iwasawa decomposition of $x_{\alpha_1}(n)$ and simplifying yields

$$I'_{F \setminus \mathfrak{o}, \mathfrak{o}}(s, t) = \int_F II(s, t; 0, m_2, 0, 0) dm_3 \int_{F \setminus \mathfrak{o}} |n|^{-51s+17} dn.$$

The first integral on the right hand side may also be obtained by killing one term in $I'_{\mathfrak{o}, \mathfrak{o}}$. So it is equal to $I'_{\mathfrak{o}, \mathfrak{o}}$ and hence to $J(1, t_2, t_1 t_2)$.

Next, lemma 13 implies that

$$I_{F \setminus \mathfrak{o}, F \setminus \mathfrak{o}}(s, t) = - \int_{F \setminus \mathfrak{o}} \int_F \int_F II(s, t; p^{-1}, m_2, m_3, n) dm_2 dm_3 dn.$$

and then lemma 14 with $\alpha = 01111110$ yields

$$I_{F \setminus \mathfrak{o}, F \setminus \mathfrak{o}}(s, t) = \int_{F \setminus \mathfrak{o}} \int_F II(s, t; p^{-1}, m_2, t_1 t_2 p^{-1}, n) dm_2 dn.$$

Plugging in the Iwasawa decomposition for $x_{\alpha_1}(n)$ and simplifying gives

$$I_{F \setminus \mathfrak{o}, F \setminus \mathfrak{o}}(s, t) = - \int_F \int_{U''} f_s(w_{\text{lng}} u x_{111111110}(m_2)) \psi_{U, t}(u) du dm_2 \int_{F \setminus \mathfrak{o}} \psi(n^{-1} p^{-2} t_1 t_2^2) dn.$$

As before, the integral on the left equals $J(1, t_2, t_1 t_2)$. Now,

$$\int_{|n|>1} |n|^{-51s+17} \psi(n^{-1} p^{-2} t_1 t_2^2) dn = \int_{|n|>1} |n|^{-51s+17} dn,$$

unless $t_1 t_2^2 \in \mathfrak{o}$, in which case

$$\int_{|n|>q} |n|^{-51s+17} \psi(n^{-1} p^{-2} t_1 t_2^2) dn = \int_{|n|>q} |n|^{-51s+17} dn,$$

while

$$\int_{|n|=q} |n|^{-51s+17} \psi(n^{-1} p^{-2} t_1 t_2^2) dn = -q^{-51s+17}.$$

Combined with the fact that

$$\int_{|n|=q} |n|^{-51s+17} dn = q^{-51s+18} - q^{-51s+17},$$

this yields the result. \square

Proposition 6.

(29)

$$I_{\mathfrak{o}, F \setminus \mathfrak{o}} = \begin{cases} 0, & |t_1| = |t_2| = 1, \\ -J(p, p^{-1} t_2, p^{-1} t_1 t_2) q^{-34s+14} + q^{-85s+35} J(1, p^{-2} t_2, p^{-2} t_1 t_2), & |t_2| < 1, \\ -q^{-68s+26} J(1, 1, p^{-2} t_1), & |t_2| = 1, |t_1| < 1. \end{cases}$$

Proof. This follows from the same type of arguments using the lemmas from section 5.1.1. We omit the details. \square

Assembling the pieces we obtain the theorem. \square

5.2. Second Reduction.

Proposition 7. *Let f be the normalized spherical vector in the induced representation attached to the character*

$$\prod_{i=1}^8 \alpha_i^\vee(t_i) \mapsto \prod_{i=1}^8 |t_i|^{s_i}, \quad \text{with}$$

$$[s_1, \dots, s_8] = [17s - 6, 17s - 6, 17s - 6, -34s + 14, 17s - 6, -17s + 7, 17s - 6, 17s - 5],$$

and let

$$J_0(b, c) := \int_{U_{w[57687]}} f(w[57687]u) \psi(bu_{00001110} + cu_{00000111}) du.$$

Then

$$J(a, b, c) = \frac{\zeta(17s - 6)\zeta(17s - 7)}{\zeta(17s - 4)\zeta(17s - 3)\zeta(17s - 2)\zeta(17s)\zeta(34s - 10)} (1 - |a|^{17s-7} q^{-17s+7}) J_0(b, c).$$

Proof. First, write $w[243154234565423145765423187] = w''w'$ with $w'' = w[243154234654237654]$, $w' = w[131257687]$. We have

(30)

$$J(a, b, c) = \int_{U_{w'}} (M(s, w'') f_s)(w'u) \psi(u_{00100000} + au_{01000000} + u_{10000000} + bu_{00001110} + cu_{00000111}) du,$$

where

$$M(s, w'')f_s(g) = \int_{U_{w''}} f_s(w''ug) du$$

is the standard intertwining operator. By the Gindikin-Karpelevich formula we have

$$M(s, w'')f_s = \frac{\zeta(17s-6)^4\zeta(34s-13)}{\zeta(17s-4)\zeta(17s-3)\zeta(17s-2)\zeta(17s)\zeta(34s-10)}f.$$

Next, observe that w' and $U_{w'}$ are contained in the standard Levi subgroup of E_8 which has derived group isomorphic to $SL_2 \times SL_3 \times SL_5$. One may factor w' as $w' = w[131]w[2]w[57687]$, and also factor the integral in (30) into a product of 3 simpler integrals corresponding to the three components of the Levi and the three factors of w' . The integrals corresponding to $w[131]$ and $w[2]$ are Jacquet integrals, equalling $\zeta(17s-6)^{-2}\zeta(34s-13)^{-1}$ and $\frac{\zeta(17s-7)}{\zeta(17s-6)}(1 - |a|^{17s-7}q^{-17s+7})$, respectively. The proposition follows. \square

5.3. An SL_5 period. From Theorem 4 and Proposition 7. the computation of $I(s, t)$, is reduced to the computation of an integral $J_0(b, c)$ over a unipotent subgroup of the copy of SL_5 in E_8 which is generated by $x_{\pm\alpha_5}$; $x_{\pm\alpha_6}$; $x_{\pm\alpha_7}$ and $x_{\pm\alpha_8}$. In this section we compute $J_0(b, c)$.

Proposition 8. *The function $J_0(b, c)$ depends only on the \mathfrak{p} -adic valuations of b and c . Take integers B and C with $B \leq C$, and take E equal to either an integer or ∞ . Define $J_2(B, C, E)$ to be*

$$\frac{(1-xq^6)(1-(xq^7)^{B+1})}{(1-xq^7)^2(1-x^2q^{13})} \begin{cases} (1-x^2q^{12})(1-xq^6) - (1-xq^5)(1-x^2q^{13})(xq^7)^{C+1} & \min(B, C, E) \geq 0 \\ \quad + (1-q^{-1})(xq^6)(1-xq^7)(x^2q^{13})^{C+1}, & E \geq C, \\ (1-x^2q^{12})(1-xq^6)(1-(x^2q^{13})^{E+1}) & \min(B, C, E) \geq 0, \\ \quad - (xq^7)^{C+1}(1-xq^5)(1-x^2q^{13})(1-(xq^6)^{E+1}), & E < C, \\ 0 & \min(B, C, E) < 0. \end{cases}$$

where $x = q^{-s}$. Then for all $b, c \in F$ with $|b| = q^{-B}$ and $|c| = q^{-C}$,

(31)

$$J_0(b, c) = J_2(B, C, \infty) + (1-q^{-1}) \sum_{\ell=1}^B J_2(B-\ell, C-\ell, \infty)(xq^8)^\ell + (1-q^{-1}) \sum_{k=1}^B (x^2q^{13})^k J_2(B-k, C, \infty) \\ + (1-q^{-1})^2 \sum_{k=1}^B \left[\sum_{\ell=0}^{k-1} J_2(B-k, C, C-k+\ell)q^{-\ell} + \sum_{\ell=1}^{B-k} J_2(B-k-\ell, C-\ell, C-k-\ell)(xq^8)^\ell \right].$$

Proof. Define

$$n^-(x_1, x_2, x_3, x_4, x_5) := x_{\alpha_5+\alpha_6+\alpha_7}(x_5)x_{\alpha_6+\alpha_7+\alpha_8}(x_4)x_{\alpha_7+\alpha_8}(x_3)x_{\alpha_6+\alpha_7}(x_2)x_{\alpha_7}(x_1) \in E_8.$$

This gives an explicit parametrization of the group $U_{w[57687]}$. Next, define

$$\hat{J}_1(b, c, e) = \int_{F^4} f_\chi(n^-(0, x_2, x_3, x_4, x_5))\psi(bx_5 + cx_4 + ex_2x_3) dx$$

$$\begin{aligned}\hat{J}_2(b, c, e) &= \int_{F^4} f_\chi(n^-(0, 0, x_3, x_4, x_5)) \psi(bx_5 + cx_4 + ex_3) dx \\ \hat{J}_4(c, e) &= \int_{F^2} f_\chi(n^-(0, 0, x_3, x_4, 0)) \psi(cx_4 + ex_3) dx.\end{aligned}$$

Then by plugging in the Iwasawa decompositions of $x_{\alpha_7}(x_1)$ and then $x_{\alpha_6+\alpha_7}(x_2)$, one finds that

$$(32) \quad \begin{aligned}J_0(b, c) &= \hat{J}_1(b, c, 0) + \int_{F \setminus \mathfrak{o}} \hat{J}_1(bx_1, b, bx_1) |x_1|^{-34s+13} dx_1 \\ \hat{J}_1(b, c, e) &= \int_{\mathfrak{o}} \hat{J}_2(b, c, x_2) dx_2 + \int_{F \setminus \mathfrak{o}} \hat{J}_2(bx_2, cx_2, ex_2) |x_2|^{-17s+7} dx_2.\end{aligned}$$

Moreover, if $1_{\mathfrak{o}}$ is the characteristic function of \mathfrak{o} , then

$$(33) \quad \hat{J}_2(b, c, e) = \frac{\zeta(17s-7)}{\zeta(17s-6)} (1 - |b|^{17s-7} q^{-17s+7}) \hat{J}_4(c, e) \cdot 1_{\mathfrak{o}}(b).$$

(The integration in x_5 amounts to an SL_2 Jacquet integral.) Likewise

$$\begin{aligned}\hat{J}_4(c, e) &= \frac{\zeta(17s-7)}{\zeta(17s-6)} \left(1_{\mathfrak{o}}(c) \int_{\mathfrak{o}} (1 - |c|^{17s-7} q^{-17s+7}) \psi(ex_3) dx_3 \right. \\ &\quad \left. + \int_{F \setminus \mathfrak{o}} 1_{\mathfrak{o}}(cx_3) (1 - |cx_3|^{17s-7} q^{-17s+7}) \psi(ex_3) |x_3|^{-34s+12} dx_3 \right).\end{aligned}$$

At this point it is clear that \hat{J}_4 , and hence all of the other integrals, depend only on the absolute values, or, equivalently, \mathfrak{p} -adic valuations, of their arguments. Introducing the notation $x := q^{-17s}$, we have

$$\hat{J}_4(p^C, p^E) = 1_{\mathfrak{o}}(c) 1_{\mathfrak{o}}(e) \frac{1 - xq^6}{1 - xq^7} \left((1 - (xq^7)^{C+1}) + \sum_{m=1}^C (1 - (xq^7)^{C-m+1}) (x^2 q^{12})^m \int_{|x_3|=q^m} \psi(ex_3) dx_3 \right).$$

Denote this quantity by $J_4(C, E)$. Recall that for any $e \in F$ with $|e| = q^{-E}$,

$$\int_{|x_3|=q^m} \psi(ex_3) dx_3 = \begin{cases} (1 - q^{-1})q^m, & m \leq E, \\ -q^E, & m = E + 1, \\ 0, & m > E + 1. \end{cases}$$

It follows that $J_4(C, E) = 0$ if either C or E is negative, and that otherwise

$$J_4(C, E) = \frac{1}{(1 - xq^7)(1 - x^2 q^{13})} \begin{cases} (1 - x^2 q^{12})(1 - xq^6) - (1 - xq^5)(1 - x^2 q^{13})(xq^7)^{C+1} \\ \quad + (1 - q^{-1})(xq^6)(1 - xq^7)(x^2 q^{13})^{C+1}, & E \geq C, \\ (1 - x^2 q^{12})(1 - xq^6)(1 - (x^2 q^{13})^{E+1}) \\ \quad - (xq^7)^{C+1}(1 - xq^5)(1 - x^2 q^{13})(1 - (xq^6)^{E+1}), & E < C. \end{cases}$$

It follows from (33) that

$$\hat{J}_2(b, c, e) = \frac{1 - xq^6}{1 - xq^7} (1 - (xq^7)^{B+1}) J_4(C, E) 1_{\mathfrak{o}}(b) = J_2(B, C, E),$$

whenever $b, c \in F^\times, e \in F$ have valuations B, C, E respectively (with the convention that the valuation of 0 is ∞). Plugging in to (32), and using the fact that $B \leq C$ and that the volume of $\{y \in F : |y| = q^k\}$ is $q^k(1 - q^{-1})$ for each $k \in \mathbb{Z}$, gives the result. \square

Now, take $X = (X_1, \dots, X_6)$ to be a sextuple of indeterminates, and consider the ring $R_1 := \mathbb{C}(x, q)[X]$ of polynomials in X with coefficients in the field $\mathbb{C}(x, q)$ of rational functions. We define two elements of this ring by

$$\begin{aligned} \mathcal{J}_2^1(X) &= (1 - X_1 X_2^7) \left(\begin{aligned} &(1 - x^2 q^{12})(1 - x q^6) - (1 - x q^5)(1 - x^2 q^{13}) X_3 X_4^7 \\ &+ (1 - q^{-1})(x q^6)(1 - x q^7) X_3^2 X_4^{13} \end{aligned} \right) \\ \mathcal{J}_2^2(X) &= (1 - X_1 X_2^7) \left((1 - x^2 q^{12})(1 - x q^6)(1 - X_5^2 X_6^{13}) - X_3 X_4^7 (1 - x q^5)(1 - x^2 q^{13})(1 - X_5 X_6^6) \right), \end{aligned}$$

then we have

$$J_2(B, C, E) = \frac{(1 - x q^6)}{(1 - x q^7)^2 (1 - x^2 q^{13})} \begin{cases} \mathcal{J}_2^1(x^{B+1}, q^{B+1}, x^{C+1}, q^{C+1}, x^{E+1}, q^{E+1}), & E \geq C, \\ \mathcal{J}_2^2(x^{B+1}, q^{B+1}, x^{C+1}, q^{C+1}, x^{E+1}, q^{E+1}), & E < C. \end{cases}$$

Let R_2 be the ring of Laurent polynomials $\mathbb{C}(x, q)[X_1, X_2, X_3, X_4, X_1^{-1}, X_2^{-1}]$. Then $\mathcal{J}_2^1 \in R_2$, and each of the four summations above corresponds to a linear operator $R_1 \rightarrow R_2$. For example,

$$\begin{aligned} &\sum_{\ell=1}^B (x^{n_1} q^{n_2})^{B+1-\ell} (x^{n_3} q^{n_4})^{C+1-\ell} (x q^8)^\ell \\ &= (x^{n_1} q^{n_2})^{B+1} (x^{n_3} q^{n_4})^{C+1} \frac{x^{1-n_1-n_3} q^{8-n_2-n_4} - (x^{1-n_1-n_3} q^{8-n_2-n_4})^{B+1}}{1 - x^{1-n_1-n_3} q^{8-n_2-n_4}}, \end{aligned}$$

for all $n_1, \dots, n_4 \in \mathbb{Z}$ with $(n_1 + n_3, n_2 + n_4) \neq (1, 8)$. It follows that

$$\sum_{\ell=1}^B J_2(B - \ell, C - \ell, \infty) = [T_1 \cdot \mathcal{J}_2^1](x^{B+1}, q^{B+1}, x^{C+1}, q^{C+1}),$$

where T_1 is the $\mathbb{C}(x, q)$ -linear map $R_1 \rightarrow R_2$ defined on monomials by

$$T_1 \left(\prod_{i=1}^6 X_i^{n_i} \right) = \prod_{i=1}^4 X_i^{n_i} (x^{1-n_1-n_3} q^{8-n_2-n_4} - X_1^{1-n_1-n_3} X_2^{8-n_2-n_4}).$$

In similar fashion, we can define operators corresponding to the other three summations in (31). Specifically, if

$$\begin{aligned} T_2 \left(\prod_{i=1}^6 X_i^{n_i} \right) &= \prod_{i=1}^4 X_i^{n_i} \frac{x^{2-n_1} q^{13-n_2} - X_1^{2-n_1} X_2^{13-n_2}}{1 - x^{2-n_1} q^{13-n_2}}, \\ T_3 \left(\prod_{i=1}^6 X_i^{n_i} \right) &= \prod_{i=1}^4 X_i^{n_i} \cdot X_3^{n_5} X_4^{n_6} \left[\frac{x^{2-n_1-n_5} q^{14-n_2-n_6}}{(1 - x^{2-n_1-n_5} q^{14-n_2-n_6})(1 - x^{2-n_1} q^{13-n_2})} \right. \\ &\quad \left. + \frac{1}{1 - x^{n_5} q^{n_6-1}} \times \left(\frac{-X_1^{2-n_1-n_5} X_2^{14-n_2-n_6}}{1 - x^{2-n_1-n_5} q^{14-n_2-n_6}} + \frac{X_1^{2-n_1} X_2^{13-n_2}}{1 - x^{2-n_1} q^{13-n_2}} \right) \right] \end{aligned}$$

$$T_4 \left(\prod_{i=1}^6 X_i^{n_i} \right) = \prod_{i=1}^4 X_i^{n_i} \cdot X_3^{n_5} X_4^{n_6} \left(\frac{x^{3-2n_1-n_3-2n_5} q^{22-2n_2-n_4-2n_6}}{(1-x^{1-n_1-n_3-n_5} q^{8-n_2-n_4-n_6})(1-x^{2-n_1-n_5} q^{14-n_2-n_6})} \right. \\ \left. - \frac{x^{1+n_3} q^{6+n_4} X_1^{1-n_1-n_3-n_5} X_2^{8-n_2-n_4-n_6}}{(1-x^{1-n_1-n_3-n_5} q^{8-n_2-n_4-n_6})(1-x^{1+n_3} q^{6+n_4})} + \frac{X_1^{2-n_1-n_5} X_2^{14-n_2-n_6}}{(1-x^{2-n_1-n_5} q^{14-n_2-n_6})(1-x^{1+n_3} q^{6+n_4})} \right).$$

Then $J_0(p^B, p^C) = \mathcal{J}_0(x^{B+1}, q^{B+1}, x^{C+1}, q^{C+1})$, where

$$\mathcal{J}_0 = \frac{1-xq^6}{(1-xq^7)^2(1-x^2q^{13})} [\mathcal{J}_2^1 + (1-q^{-1})(T_1+T_2) \cdot \mathcal{J}_2^1 + (1-q^{-1})^2(T_3+T_4) \cdot \mathcal{J}_2^1].$$

Proposition 9. *The value of \mathcal{J}_0 is given by*

$$\frac{(1-xq^6)(1-x^2q^{12})}{(1+xq^7)^3(1-x^2q^{13})} \left(\frac{(1-xq^6)(1-x^2q^{13})}{(1-q^8x)} - \frac{X_1X_2^8(1-xq^5)(1-x^2q^{14})}{(1-xq^8)} + (1-q^{-1})xq^6X_1^2X_2^{14} \right) \\ + X_3X_4^7(1-q^5x) \left(-(1+q^6x)(1-q^7x)X_2q^{-1} + (1-q^{13}x^2)q^{-1}X_1X_2^7 - (1-q^{-1})q^6xX_1^2X_2^{14} \right) \\ + X_3^2X_4^{13} \frac{(1-q^{-1})}{xq^7(1-q^8x)} ((1-q^8x)(1+x^2q^{12}-x^2q^{13}-x^3q^{19})X_2 \\ + xq^8(1-xq^4-xq^6+x^2q^{11}+x^2q^{12}-x^2q^{13})X_1X_2^8)$$

Proof. Straightforward calculations give each of the components of the sum. We record the results:

$$[T_1 \cdot \mathcal{J}_2^1] = (1-xq^6)(1-x^2q^{12}) \left(\frac{xq^8}{1-xq^8} - \frac{q}{1-q}X_1X_2^7 + \frac{q(1-xq^7)}{(1-xq^8)(1-q)}X_1X_2^8 \right) \\ - (1-xq^5)(1-x^2q^{13}) \left(\frac{q}{1-q} + \frac{1}{1-xq^6}X_1X_2^7 - \frac{1-xq^7}{(1-q)(1-xq^6)}X_2 \right) X_3X_4^7 \\ + (1-q^{-1})(1-xq^7)(xq^6) \left(\frac{-1}{1-xq^5} + \frac{1}{1-x^2q^{12}}X_1X_2^7 + \frac{xq^5(1-xq^7)}{(1-xq^5)(1-x^2q^{12})}X_1^{-1}X_2^{-5} \right) X_3^2X_4^{13}$$

$$[T_2 \cdot \mathcal{J}_2^1] = [T_2 \cdot (1-X_1X_2^7)] \mathcal{J}_4^1 = \left(\frac{x^2q^{13}}{1-x^2q^{13}} - \frac{xq^6X_1X_2^7}{1-xq^6} + \frac{xq^6X_1^2X_2^{13}(1-xq^7)}{(1-xq^6)(1-x^2q^{13})} \right) \mathcal{J}_4^1$$

$$[T_3 \cdot \mathcal{J}_2^2] = ((1-xq^6)(1-x^2q^{12}) - (1-xq^5)(1-x^2q^{13})X_3X_4^7) \cdot [T_3 \cdot (1-X_1X_2^7)] + c_2(X_1, X_2)X_3^2X_4^{13},$$

where

$$T_3 \cdot (1-X_1X_2^7) = \frac{x^2q^{14}}{(1-x^2q^{14})(1-x^2q^{13})} - \frac{X_1X_2^7xq^7}{(1-xq^7)(1-xq^6)} \\ - \frac{X_1^2X_2^{13}xq^6(1-xq^7)}{(1-q^{-1})(1-xq^6)(1-x^2q^{13})} + \frac{X_1^2X_2^{14}xq^7}{(1-q^{-1})(1-x^2q^{14})}$$

$$\begin{aligned}
c_2 = (1 - xq^5)(1 - x^2q^{13}) & \left[\frac{xq^8}{(1 - xq^8)(1 - x^2q^{13})} - \frac{X_1X_2^7q}{(1 - xq^6)(1 - q)} \right. \\
& + \frac{X_1X_2^8q}{(1 - xq^5)(1 - xq^8)} - \frac{X_1X_2^{13}xq^6(1 - xq^7)}{(1 - xq^6)(1 - x^2q^{13})(1 - xq^5)} \Big] \\
& - (1 - xq^6)(1 - x^2q^{12}) \left[\frac{q}{(1 - q)(1 - x^2q^{13})} - \frac{X_1X_2^7x^{-1}q^{-6}}{(1 - xq^6)(1 - x^{-1}q^{-6})} \right. \\
& + \frac{X_2x^{-1}q^{-6}}{(1 - x^2q^{12})(1 - q)} - \frac{X_1^2X_2^{13}xq^6(1 - xq^7)}{(1 - xq^6)(1 - x^2q^{13})(1 - x^2q^{12})} \Big]
\end{aligned}$$

Finally

$$\begin{aligned}
[T_4 \cdot \mathcal{J}_2^2] = & (1 - xq^6)(1 - x^2q^{12})Q(0, 0, 0, 0) - (1 - xq^5)(1 - x^2q^{13})Q(1, 7, 0, 0)X_3X_4^7 \\
& - [(1 - xq^6)(1 - x^2q^{12})Q(2, 13, 0, 0) - (1 - xq^5)(1 - x^2q^{13})Q(1, 7, 1, 6)]X_3^2X_4^{13}.
\end{aligned}$$

where

$$\begin{aligned}
Q(n_3, n_4, n_5, n_6) = & \frac{x^{3-n_3-2n_5}q^{22-n_4-2n_6}}{(1 - x^{2-n_5}q^{14-n_6})(1 - x^{1-n_3-n_5}q^{8-n_4-n_6})} \\
& - \frac{X_1X_2^7x^{1-n_3-2n_5}q^{8-n_4-2n_6}}{(1 - x^{1-n_5}q^{7-n_6})(1 - x^{-n_3-n_5}q^{1-n_4-n_6})} \\
& - \frac{x^{1-n_5}q^{7-n_6}X_1^{2-n_5}X_2^{14-n_6}(1 - q^7x)}{(1 - x^{n_3+1}q^{n_4+6})(1 - x^{1-n_5}q^{7-n_6})(1 - x^{2-n_5}q^{14-n_6})} \\
& + \frac{x^{1-n_5}q^{7-n_6}X_1^{1-n_3-n_5}X_2^{8-n_4-n_6}(1 - q^7x)}{(1 - x^{n_3+1}q^{n_4+6})(1 - x^{-n_3-n_5}q^{1-n_4-n_6})(1 - x^{1-n_3-n_5}q^{8-n_4-n_6})}.
\end{aligned}$$

Totalling up the contributions, multiplying by $\frac{(1-xq^6)}{(1-xq^7)^2(1-x^2q^{13})}$, and simplifying gives the result. \square

5.4. Final calculation of $I(s, t)$. In this section we complete the proof of theorem 3. First, by proposition 7, $J(a, b, c) = P_0(x, q)(1 - (xq^7)^{A+1})J_0(b, c)$, where

$$P_0(x, q) = \frac{(1 - x)(1 - xq^2)(1 - xq^3)(1 - xq^4)(1 - x^2q^{10})}{(1 - xq^6)(1 - xq^7)},$$

and by theorem 4

$$I(s, t) = \begin{cases} J(1, 1, 1)(1 + x^3q^{18}), & t_1, t_2 \in \mathfrak{o}^\times \\ J(1, 1, t_1) - x^4q^{26}J(1, 1, \frac{t_1}{p^2}), & t_2 \in \mathfrak{o}^\times, t_1 \notin \mathfrak{o}^\times \\ J(1, t_2, t_1t_2) - x^2q^{14}J(p, \frac{t_2}{p}, \frac{t_1t_2}{p}) + x^5q^{35}J(1, \frac{t_2}{p^2}, \frac{t_1t_2}{p^2}), & t_2 \notin \mathfrak{o}^\times. \end{cases}$$

5.4.1. *Case 1:* $t_2 \notin \mathfrak{o}^\times$. First, assume $t_2 \notin \mathfrak{o}^\times$, define B and C in terms of t_1, t_2 by $|t_2| = q^{-B}$, $|t_1 t_2| = q^{-C}$. Then

$$\begin{aligned} I(s, t) &= J(1, t_2, t_1 t_2) - x^2 q^{14} J(p, \frac{t_2}{p}, \frac{t_1 t_2}{p}) + x^5 q^{35} J(1, \frac{t_2}{p^2}, \frac{t_1 t_2}{p^2}) \\ &= (1 - xq^7) \left(J_0(t_2, t_1 t_2) - x^2 q^{14} (1 + xq^7) J_0(\frac{t_2}{p}, \frac{t_1 t_2}{p}) + x^5 q^{35} J_0(\frac{t_2}{p^2}, \frac{t_1 t_2}{p^2}) \right) \\ &= (1 - xq^7) [T_0 \cdot \mathcal{J}_0](x^{m+1}, q^{m+1}, x^{n+m+1}, q^{n+m+1}), \end{aligned}$$

where n and m are the \mathfrak{p} -adic valuations of t_1 and t_2 , respectively, and T_0 is a linear operator on the ring R_2 defined by

$$T_0 \left(\prod_{i=1}^4 X_i^{n_i} \right) = \prod_{i=1}^4 X_i^{n_i} (1 - x^{2-n_1-n_3} q^{14-n_2-n_4}) (1 - x^{3-n_1-n_3} q^{21-n_2-n_4}).$$

Applying the operator T_0 to \mathcal{J}_0 as computed in the previous section gives

$$[T_0 \cdot \mathcal{J}_0] = \frac{(1 - xq^6)(1 - x^2 q^{12})}{(1 - xq^7)(1 - xq^8)} \left[\begin{aligned} &(1 - xq^6)(1 - x^3 q^{21}) - (1 - xq^6)(1 - xq^8) X_3 X_4^7 \\ &- q^{-1} X_2 X_3 X_4^7 (1 - xq^5)(1 - xq^8) \end{aligned} \right].$$

Multiplying by $(1 - xq^7)P_0(x, q) = (1 - xq^6)(1 - x^2 q^{12})Z(x, q)$, and plugging in $X_2 = q^{m+1}$, $X_3 = x^{n+m+1}$ and $X_4 = q^{n+m+1}$ gives the proof in this case.

5.4.2. *Case 2:* $t_2 \in \mathfrak{o}^\times$. In this case the summations corresponding to T_1, T_3 and T_4 are empty sums. Hence these terms will specialize to zero in that case. Moreover

$$(\mathcal{J}_2^1 + (1 - q^{-1})T_2 \cdot \mathcal{J}_2^1) = \frac{1}{(1 - xq^6)(1 - x^2 q^{13})} \mathcal{J}_4^1(x, q, X_1, X_2) \mathcal{J}_4^1(x, q, X_3, X_4).$$

The condition $t_2 \in \mathfrak{o}^\times$ translates to $X_1 = x, X_2 = q$, and

$$\mathcal{J}_4^1(x, q, X_1, X_2) = (1 - xq^6)(1 - xq^7)(1 - x^2 q^{13}),$$

so we obtain $\mathcal{J}_0 = \frac{1-xq^6}{(1-xq^7)(1-x^2 q^{13})} \mathcal{J}_4^1(x, q, X_3, X_4)$ in this case. Now,

$$I = P_0(x, q)(1 - xq^7) \begin{cases} (1 + x^3 q^{18}) \mathcal{J}_0(x, q, x, q), & t_1 \in \mathfrak{o}^\times, \\ \mathcal{J}_0(x, q, X_3, X_4) - x^4 q^{26} \mathcal{J}_0(x, q, X_3 x^{-2}, X_4 q^{-2}), & t_2 \notin \mathfrak{o}^\times, \end{cases}$$

Simplifying

$$\begin{aligned} &\mathcal{J}_0(x, q, X_3, X_4) - x^4 q^{26} \mathcal{J}_0(x, q, X_3 x^{-2}, X_4 q^{-2}) \\ &= \frac{1 - xq^6}{(1 - xq^7)(1 - x^2 q^{13})} ((1 - xq^6)(1 - x^2 q^{12})(1 - x^4 q^{26}) - (1 - xq^5)(1 - x^2 q^{13})(1 - x^2 q^{12}) X_3 X_4^7). \end{aligned}$$

If $X_3 = x$, and $x = q$, this coincides with $(1 + x^3 q^{18}) \mathcal{J}_0(x, q, x, q)$. It follows that the case $t_1 \in \mathfrak{o}^\times$ does not need to be handled separately. After incorporating $(1 - xq^7)P_0(x, q)$, one has only to check that

$$I_0(n, 0; x, q) = (1 - xq^8) ((1 - xq^6)(1 + x^2 q^{13}) - (1 - xq^5) x^{n+1} q^{7n+7}),$$

and this is straightforward.

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